# NOTES FOR 23 OCT (THURSDAY)

## 1. Recap

(1) Proved a special case of the uniformisation theorem using the method of continuity (solving a linear PDE, assume WLOG $K_0 < 0$).
(2) Relationship with Riemann surfaces.

## 2. The uniformisation theorem for Genus $\geq 2$ : A variational method

If $g = e^{-f}g_0$ and we want $K(g) = K$, then $\Delta f = Ke^{-f} - K_0$. It is a semilinear PDE. Let $K = -1$. Let the area of $g_0$ be normalised to be 1. One can prove existence by minimising a certain functional. This approach is there in Kazdan-Warner ("Curvature function for compact two manifolds".)
Let $E[f] = \frac{1}{2}\int_M |\nabla f|^2 - \int_M K_0 f$. We want to minimise this functional over $H^1$ with the constraint $\displaystyle\int_M -e^{-f} = 2\pi\chi(M)$. Why is this problem even sensible ? Zeroethly, ignore the constraint for the next few minutes. Firstly, it is obvious (Cauchy-Schwarz) that $E(f)$ is finite for $H^1$ functions. Secondly, it is bounded below by C-S again. Now we come to the pesky constraint, i.e., suppose you have a sequence $u_i \to u$ in $H^1$ and $u_i$ satisfy the constraint, then what about $u$?
(From now onwards, we pass to subsequences without mentioning the same, like evil chumps.)
Firstly, there is a constant $C$ (independent of $p$) such that $\|w\|_{L^p} \leq C\sqrt{p}\|w\|_{H^1}$. Indeed, locally, if $w$ is compactly supported, then $w(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \nabla_{euc} w(x-y).\frac{y}{|y|^2}dy$ (by the divergence theorem) from which Young's inequality ($\|f * g\|_{L^p} \leq \|f\|_{L^2}\|g\|_{L^{2p/(p+2)}}$ for $p \geq 2$) and a partition-of-unity give the result.
A power series expansion, the above inequality, and Poincaré 's inequality (A HW exercise asks you to prove the Poincaré inequality : There exists a constant $C$ such that $C\|\nabla f\|_{L^2} \geq \|f - \fint f\|_{L^2}$) show the Moser-Trudinger inequality : $\int_M e^{\beta u^2}dA \leq \gamma$ for some positive $\beta, \gamma$ and all $\|u\|_{H^1} \leq 1$.
By AM-GM, for any $\alpha > 0$, $\int_M e^{\alpha|u|} \leq C' \exp\left(\alpha|\fint u| + C\alpha^2\|\nabla u\|^2_{L^2}\right)$.
Using $|e^t - 1| \leq |t|e^{|t|}$ and this inequality, we see that if $u_j \to u$ weakly in $H^1$, then passing to a subsequence, $e^{u_j} \to e^u$ strongly in $L^2$.Therefore, the constraint is met by $u$.

We now prove a better lower bound on $E(f)$. Indeed, wLog we can assume that $K_0 < 0$ by solving a linear equation $\Delta f_0 + K_{0,old} = 2\pi\chi(M) = K_{0,new}e^{-f_0}$. Since $x \to e^{-x}$ is convex, by Jensen's inequality $2\pi|\chi(M)| = \fint e^{-f} \geq e^{-\fint f}$ and hence $\fint f \geq -C$.
Putting the above together, we see that $E(f)$ is bounded below (in a coercive manner). Indeed, $E(f) = \frac{1}{2}\int_M |\nabla(f - \fint f)|^2 - \int_M K_0(f - \fint f) + 2\pi|\chi(M)|\fint f \geq -C + \frac{1}{C}\|f - \fint f\|^2_{H^1}$. Let $f = u + \fint f$. So $\fint u = 0$ and hence $\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}$. So there is a sequence $f_n \in H^1$ (satisfying the constraint) such that $E(f_n) \to \inf E$. By coercivity, $u_n$ weakly converges to $u$ in $H^1$. By the M-T arguments, $e^{-ku_n} \to e^{-ku}$ for every $k$ strongly in $L^2$. Solving for $\fint f_n$ from the constraint, $\fint f_n = -\ln\left(\frac{2\pi|\chi(M)|}{\int e^{-u_n}}\right)$ which converges to a number $A$. Define $f = A + u$. Hence $A = \fint f$ and it satisfies the constraint. Since $f_n \to f$ strongly in $L^2$ and $\|f\|_{H^1} \leq \liminf \|f_n\|_{H^1}$, we see that $\|\nabla f\|_{L^2} \leq \liminf \|\nabla f_n\|_{L^2}$. Hence, $E(f) = \inf(E)$

and $E(u) = \inf E$ where $f = \fint f + u$.
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