NOTES FOR 26 AUG (TUESDAY)

1. Recap

(1) Proved a few properties of Fredholm operators.
(2) Proved elliptic regularity and properties about the formal adjoint.
(3) Motivated the need for a Riemannian metric to talk about the Laplace equation.

2. Riemannian manifolds and metrics on vector bundles

Recall that a smooth vector bundle V over a smooth manifold M is a "smoothly varying collection of vector spaces parametrised by M", i.e., locally, $V \simeq U \times \mathbb{R}^r$ (where instead of $\mathbb{R}$, we can also have $\mathbb{C}$ - such a beast is a complex vector bundle) via a trivialisation, i.e., a collection of smooth sections $e_1, \ldots, e_r : U \subset M \to V$ such that $e_1(p), \ldots, e_r(p)$ form a basis for V_p at all $p \in U$. Equivalently, a vector bundle is simply a collection $(U_\alpha, g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r))$ satisfying $g_{\alpha\alpha} = Id, g_{\alpha\beta} = g_{\beta\alpha}^{-1}, g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = Id$. Fundamental examples of vector bundles are the tangent bundle TM, the cotangent bundle T^*M, and the bundles of differential forms $\Omega^k(M)$. These bundles can be defined using transition functions. A smooth section $s : M \to V$ is a smooth function satisfying $\pi \circ s = Id$. We can construct tensor products and direct sums of bundles.

A metric g on a vector bundle V over M is a smooth section of $V^* \otimes V^*$ such that on each fibre it is symmetric and positive-definite. In other words, suppose e_i is a trivialisation of V over U and e^{i*} the dual trivialisation of V^* over U, then[1] $g(p) = g_{ij}(p)e^{i*} \otimes e^{j*}$ where $g_{ij} : U \subset M \to GL(r, \mathbb{R})$ is a smooth matrix-valued function to symmetric positive-definite matrices. So a metric is simply a smoothly varying collection of dot products, one for each fibre. Using a partition-of-unity one can prove the following result.

Theorem 2.1. *Every rank-r real vector bundle V over a manifold M admits a smooth metric g.*

In the special case when $V = TM$, the metric is called a Riemannian metric on M. If (x, U) is a coordinate chart, then $g(x) = g_{ij}(x)dx^i \otimes dx^j$. By symmetry, $g_{ij} = g_{ji}$. Moreover, g is a positive definite matrix. If one changes coordinates to y^μ then $g_{\mu\nu} = g_{ij}\frac{\partial x^i}{\partial y^\mu}\frac{\partial x^j}{\partial y^\nu}$. Given a metric g on TM, we get one on T^*M given by $g^* = g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$. Note that $g_{ik}g^{kj} = \delta_i^j$.

If M is oriented, supposing (x, U) is an oriented coordinate chart, then $vol = \sqrt{\det(g_{ij})}dx^1 \wedge dx^2 \ldots dx^m$ is a well-defined top form. Indeed, if we changes coordinates, it transforms correctly as seen in the linear algebra above. This form is called the "volume" form of the metric.

Here are examples :

(1) Euclidean space $\mathbb{R}^n, g_{Euc} = \sum dx^i \otimes dx^i$. This is the usual metric. The length of a tangent vector v is $\sum (v^i)^2$.
(2) If we take the same Euclidean space $\mathbb{R}^2$ and use polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$, then $dx = dr\cos(\theta) - r\sin(\theta)d\theta$, $dy = dr\sin(\theta) + r\cos(\theta)d\theta$. Thus, $g_{Euc} = dr \otimes dr + r^2 d\theta \otimes d\theta$.
(3) The circle $S^1 : g = d\theta \otimes d\theta$.

[1] We will be using the Einstein summation convention. Repeated indices are understood to be summed over.

(4) If M, g_M, N, g_N are two Riemannian manifolds, then $M \times N, g_M \times g_N$ given by $g_M \times g_N(v_M \oplus v_N, w_M \oplus w_N) = g_M(v_M, w_M) + g_N(v_N, w_N)$. This gives a metric on the n-torus using the circle metric.
(5) The Hyperbolic metric $\mathbb{H}^m, g_{Hyp} : g_{Hyp} = \frac{\sum dx^i \otimes dx^i}{(x^m)^2}$.

Recall the definition of an induced metric

Definition 2.2. If g is a metric on M and $S \subset M$ is an embedded submanifold, then g induces a metric $g|_S$ on S given by $g_p|_S(v_S, w_S) = g_p(i_* v_S, i_* w_S)$.

(1) $S^2 \subset \mathbb{R}^3$. First write the metric in $\mathbb{R}^3$ in spherical coordinates $z = r\cos(\theta)$, $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$. Thus, $g_{Euc} = dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2(\theta) d\phi \otimes d\phi$. Now when we restrict to the unit sphere, the tangent vectors do not include $\frac{\partial}{\partial r}$. Thus, $g_{Sphere} = d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi$
(2) Suppose $z = f(x, y)$ is the graph of a function, then $g_{Induced} = dx \otimes dx + dy \otimes dy + (\frac{\partial f}{\partial x})^2 dx \otimes dx + (\frac{\partial f}{\partial y})^2 dy \otimes dy + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}(dx \otimes dy + dy \otimes dx)$.

Now we write down the volume forms of most of the above examples :

(1) $vol_{Euc} = dx^1 \wedge dx^2 \wedge \ldots dx^n$.
(2) In polar coordinates in $\mathbb{R}^2$, $vol_{Euc} = \sqrt{\det(g)} dr \wedge d\theta = r dr \wedge d\theta$.
(3) For the circle, $vol = d\theta$.

Suppose $\gamma : [0, 1] \to M$ is a smooth path. Then, define its length as $L(\gamma) = \int_0^1 \sqrt{g(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})} dt$. A continuous piecewise C^1, regular (meaning that $\gamma' \neq 0$ throughout) curve satisfying the following equation weakly is called a geodesic.

$$\frac{d^2\gamma^r}{dt^2} + \Gamma^r_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0$$
$$\Gamma^r_{ij} = g^{rl}\frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right) \tag{2.1}$$

It can be proved that every geodesic is actually smooth, can be parametrised by its arc-length, and that arc-length parametrised geodesics are precisely the critical points of the length functional. Moreover, the function $d(p, q) = \inf L(p, q)$ over all the piecewise C^1 paths joining p and q is a metric and that the topology induced by it is the same as the original topology of the manifold.

Now we note that geodesics exist locally, and that if γ is a geodesic, then so is $\gamma(ct)$. In fact, we have the following result.

Theorem 2.3. *Let $p \in M$. Then there is a neighbourhood U_o of p and a number $\epsilon_p > 0$ such that for every $q \in U$ and every tangent vector $v \in T_qM$ with $\|v\| < \epsilon_p$ there is a unique geodesic $\gamma_v : (-2, 2) \to M$ satisfying $\gamma_v(0) = q$, $\frac{d\gamma_v}{dt}(0) = v$.*

If $v \in T_qM$ is a vector for which there is a geodesic, $\gamma : [0, 1] \to M$ satisfying $\gamma(0) = q$ and $\gamma'(0) = v$ then we define $\exp_q(v) = \gamma_v(1)$. The geodesic itself can be described as $\gamma(t) = \exp_q(tv)$ (by the uniqueness theorem for ODE).