NOTES FOR 28 AUG (THURSDAY)

1. Recap

(1) Induced metrics on submanifolds and examples.
(2) Geodesics and some properties.

2. Riemannian manifolds

By the smooth dependence on parameters of an ODE, $\exp_q(v)$ depends smoothly on q and on v and defines a smooth map $\exp_q : T_qM \to M$.
Note that $(\exp_q)_{v*} : T_v(T_qM) \simeq T_qM \to T_{\exp_q(v)}M$ is its pushforward. We claim that

Theorem 2.1. $(\exp_q)_{0*} = Id$ *and hence* $\exp_q$ *is a local diffeomorphism around* $\vec{0}$.

Proof. Clearly the first statement and the inverse function theorem imply the second. Now if $v \in T_qM$, we need to obtain a curve $c(t) \in T_qM$ such that $c(0) = 0$, $c'(0) = v$, and $\frac{d\exp_q(c(t))}{dt}|_{t=0} = v$. Let $c(t) = tv$. Then $\exp_q(c(t)) = \exp_q(tv)$ which is the time-t geodesic starting at q pointing along v at $t = 0$. Thus we are done. □

In fact, we can say more.

Theorem 2.2. *Geodesics are locally length minimising. Moreover, if $p \in M$, there exists a geodesic ball $B_{\epsilon_p}(p)$ such that every two points in the ball can be connected by a unique length minimising geodesic lying in the ball and such that the exponential map is a diffeomorphism restricted to the ball. Such a ball is called a geodesically convex ball.*

Now we make a definition of a useful coordinate system.

Definition 2.3. Given $q \in M$, the coordinate system defined by $\exp_q : U \subset T_qM \to M$ is called a geodesic normal coordinate system at q (after choosing coordinates on U that is).

This set of coordinates is extremely useful. In fact,

Theorem 2.4. *There is a geodesic normal coordinate system v at p, $g_{ij}(p) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial v^k}(p) = 0$.*

Proof. Choose coordinates x^μ so that $g_{\mu\nu}(p) = \delta_{\mu\nu}$. (This can be easily accomplished by taking any coordinate system and rotating it so as to diagonalise g.) Let v^i be coordinates in T_pM. Now exp is a local diffeomorphism. So $x^\mu(v^j) = x^\mu \circ \exp(v^j)$ is a change of coordinates in a small neighbourhood.

Note that since $\exp_{0*} = Id$, $\frac{\partial x^\mu}{\partial v^j}|_{v=0} = \delta^\mu_j$. Now $\tilde{g}_{ij} = g_{\mu\nu}\frac{\partial x^\mu}{\partial v^i}\frac{\partial x^\nu}{\partial v^j}$. So it is easy to see that $\tilde{g}_{ij}(0) = \delta_{ij}$. Since the geodesics through p are linear in this coordinate system, we see that the Christoffel symbols $\tilde{\Gamma}^r_{ij}(0) = 0$. It is easy to see that if the Christoffel symbols are 0, then so are all first partial derivatives of the metric. □

More generally, any coordinate system in which the metric at p is standard upto first order is called a normal coordinate system at p.

Actually, we can prove the existence of normal coordinates in much simpler manner even without reference to geodesics.

Theorem 2.5. *There is a normal coordinate system y at p.*

Proof. Choose any coordinate system at x at p such that $x = 0$ is p. Using a linear map, we may diagonalise g at p. So without loss of generality, $\tilde{g}_{\mu\nu} = \delta_{\mu\nu} + a_{\mu\nu\alpha}x^{\alpha} + O(x^2)$. (Note that $a_{\mu\nu\alpha} = a_{\nu\mu\alpha}$.) Change the coordinates to y such that $x(y)^i = y^i + b^i_{jk}y^j y^k$ where $b^i_{jk} = b^i_{kj}$. Now

$$g_{ij} = \tilde{g}_{\mu\nu}\frac{\partial x^{\mu}}{\partial y^i}\frac{\partial x^{\nu}}{\partial y^j} = (\delta_{\mu\nu} + a_{\mu\nu\alpha}y^{\alpha} + O(y^2))(\delta^{\mu}_i + b^{\mu}_{ik}y^k)(\delta^{\nu}_j + b^{\nu}_{jk}y^k)$$

$$= \delta_{ij} + a_{ijk}y^k + (b_{ijk} + b_{jik})y^k + O(y^2) \tag{2.1}$$

So we just need to choose b so that $a_{ijk} = -b_{ijk} - b_{jik}\ \forall\ k$. So take $b = -\frac{a}{2}$. □

It is natural to ask if there is a geodesic normal coordinate system to the second order. Shockingly enough, there isn't (in general). In fact,

Theorem 2.6. *There exists a $(0,4)$ tensor (called the Riemann curvature tensor of g) which is locally $R_{\mu\nu\alpha\beta}$ such that in geodesic normal coordinates,*

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}(0)x^{\mu}x^{\nu} + O(x^3) \tag{2.2}$$

where in these coordinates, $R_{ijkl}(0) = \frac{1}{2}\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l}(0) + \frac{1}{2}\frac{\partial^2 g_{il}}{\partial x^j \partial x^k}(0) - \frac{1}{2}\frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}(0) - \frac{1}{2}\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l}(0)$. In fact, all the other terms in the Taylor expansion depend only on R and its derivatives. So there is a change of coordinates such that g is Euclidean everywhere, then, since the Euclidean coordinates are geodesically normal, the Riemann curvature tensor is identically 0.

So one can prove that one cannot draw a map of any part of Bangalore on a piece of paper such that distances are to scale, by calculating the curvature of the sphere with the metric induced from the Euclidean space. It turns out to be a non-zero tensor. We will return to curvature later on in a different way. This theorem is to show you that the notion of curvature is "forced" upon us. (It is not an artificial definition.)

3. Connections and curvature

Here are a bunch of observations / questions :

(1) In $\mathbb{R}^n$, you have the idea of a "constant" vector field. (Indeed, this is one way to prove that $\mathbb{R}^n$ is parallelizable, i.e., it has trivial tangent bundle.) So you need to able to find the directional derivative $\nabla_V X$ of any vector field along a direction V. Note that if we manage to define this concept, then $\nabla_{\gamma'(t)}X(\gamma(t)) = 0$ amounts to parallel transporting the vector field along γ.

(2) Suppose $(S, g_S) \subset (\mathbb{R}^n, Euc)$ is a submanifold with the induced metric. (Actually every Riemannian manifold is of this form by the Nash embedding theorem.) Suppose X is a tangent vector field along S. Suppose that $N_1, N_2 \ldots, N_k$ are local linearly independent unit normal vector fields on $U \subset S$ (where $k = n - dim(S)$). Assume that V is a tangent vector on S at p. How can we define the directional derivative $\nabla_V X(p)$? Clearly, the usual Euclidean directional derivative $D_V X = \frac{\partial \vec{X}}{\partial x^i}V^i$ is not the right one because it measures how fast X is changing perpendicular to S as well. So we need to project this back to S.

In other words, the "correct" way to define a directional derivative is $\nabla_V X = D_V X - \sum_{i=1}^{k}\langle D_V X, N_i\rangle_{Euc} N_i$. Now note that $\langle D_V X, N_i\rangle_{Euc} = D_V\langle X, N_i\rangle_{Euc} - \langle X, D_V N_i\rangle_{Euc} =$

$-\langle X, D_V N_i\rangle_{Euc}$. In other words,

$$\nabla_V X = D_V X - \sum \langle X, D_V N_i\rangle_{Euc} N_i = D_V X + a\ term\ linear\ in\ X. \tag{3.1}$$

It turns out (miraculously) that the linear term is related to the Christoffel symbols and the Riemann curvature tensor of g_S that we defined before. This way of defining a directional derivative is called the Levi-Civita connection. In general, a "directional derivative" on a vector bundle is called a "connection".

(3) The above definitions of directional derivative are important even for a general vector bundle. For example if we want to prove that there is a nowhere vanishing section of a certain vector bundle, ideally, we would want to take a "constant" section. But to even define that, we need to know the notion of a directional derivative.

(4) The notion of "curvature" seems to depend on one derivative of the Christoffel symbol (or alternatively, two derivatives of the metric).

The above mentioned observations force us to define a connection $\nabla_W s$ on vector bundles. It is suppose to represent how fast a section s is changing along the tangent vector W. In fact, if W is a vector field, then $\nabla_W s$ better be a section of the vector bundle itself. So, we have

Definition 3.1. Suppose V is a smooth rank-r real vector bundle (a similar definition holds for complex vector bundles) over a smooth manifold M. Suppose $\Gamma(V)$ is the (infinite-dimensional) vector space of smooth sections of V over M. Suppose X is a vector field on M. Then a connection (sometimes called an affine connection) ∇ on V is a map $\nabla_X : \Gamma(V) \to \Gamma(V)$ satisfying the following properties.

(1) *Tensoriality in* X : If s is a smooth section of V, X_1, X_2 are two vector fields, and f_1, f_2 are two smooth functions, then $\nabla_{f_1 X_1 + f_2 X_2}(s) = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s$. In other words, the value of $\nabla_X s$ at p depends only on the value of X at p but not on the derivatives of X.

(2) *Linearity in* s : If s_1, s_2 are two sections and c_1, c_2 are two real numbers, then $\nabla_X(c_1 s_1 + c_2 s_2) = c_1 \nabla_X s_1 + c_2 \nabla_X s_2$.

(3) *Leibniz rule* : If f is a smooth function and s is a section, $\nabla_X(fs) = f\nabla_X s + df(X)s = f\nabla_X + X(f)s$.

The first assumption (tensoriality in X) can be stated in another nice way : Suppose we fix s. Then the map $(X, \alpha) \to \alpha(\nabla_X s)$ is a map from $Vect\ fields \times \Gamma(V^*) \to C^\infty\ functions$ which is multilinear (over functions). It can be proved that there exists a smooth section $T_s \in \Gamma(T^*M \otimes V^{**} \simeq V)$ such that $T_s(X, \alpha) = \alpha(\nabla_X s)$.

Thus ∇ can be thought of as a map $\Gamma(V) \to \Gamma(V \otimes T^*M)$ given by $s \to \nabla s$. The space $\Gamma(V \otimes T^*M)$ is commonly called "vector-valued 1-forms". Moreover, in this framework, a connection satisfies $\nabla(fs) = df \otimes s + f\nabla s$.

Theorem 3.2. *Every vector bundle has a connection.*

Proof. Suppose M is covered by a locally finite cover U_α of trivialisation open sets for V. Suppose $T_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}^r$ is the trivialising isomorphism of bundles. There is an obvious connection ∇_α on the trivial bundle $U_\alpha \times \mathbb{R}^r$. Define $\tilde{\nabla}_\alpha s = T_\alpha^{-1} \nabla_\alpha (T_\alpha s)$ as a connection on V on the set U_α. Suppose ρ_α is a partition-of-unity subordinate to U_α.

Now, define $\nabla s = \sum_\alpha \rho_\alpha \tilde{\nabla}_\alpha s$. The meaning of this statement is "Take s, restrict it to U_α, calculate $\tilde{\nabla}_\alpha s$ as a section over U_α, multiply by ρ_α and extend it to all of M by 0 outside U_α, and sum over all α. It is a finite sum at every point because of local finiteness of the cover. Thus we have a section of $V \otimes T^*M$"

We still have to prove that ∇ is a connection. Indeed,

$$\begin{aligned} \nabla(fs) &= \sum_\alpha \rho_\alpha \tilde{\nabla}_\alpha(fs) = \sum_\alpha \rho_\alpha (df \otimes s + f\tilde{\nabla}_\alpha s) \\ &= df \otimes s \sum_\alpha \rho_\alpha + f\nabla s = df \otimes s + f\nabla s \end{aligned} \tag{3.2}$$

□