NOTES FOR 28 OCT (TUESDAY)

1. Recap

(1) Uniformisation theorem using the variational method. Last step remaining (Euler-Lagrange equations).

2. The uniformisation theorem for Genus ≥ 2 : A variational method

$\tilde{E}(u) = E(f) = \frac{1}{2}\int_M |\nabla u|^2 - \int K_0 u - 2\pi|\chi(M)| \ln\left(\frac{2\pi|\chi(M)|}{\int e^{-u}}\right)$ and $\int u = 0$. We claim that $E(u)$ is minimum over all $\int u = 0$ H^1 functions. Indeed, suppose $\tilde{E}(w) < \tilde{E}(u)$. Then define $F = -\ln\left(\frac{2\pi|\chi(M)|}{\int e^{-w}}\right) + w$. Clearly F satisfies the constraint and $E(F) < E(f)$. Thus we have a contradiction to the fact that $E(f)$ is the minimum.
Let $v \in C^\infty(M)$ with $\fint v = 0$. $\frac{dE(u+tv)}{dt}|_{t=0} = 0$. Thus $0 = \int_M \langle \nabla v, \nabla u\rangle - \int K_0 v - 2\pi|\chi(M)|\frac{\int e^{-u} v}{\int e^{-u}}$. Thus, $u \in H^1$ is a distributional solution of $\Delta u = -K_0 - 2\pi|\chi(M)|\frac{e^{-u}}{\int e^{-u}}$, i.e., f solves the desired PDE. Now we need to show that f is smooth. $e^{-f} \in L^2$ (actually L^p for all p) and hence $f \in H^2$ by elliptic regularity. By Sobolev embedding, $f \in W^{1,p}$ for all $p < \infty$. Hence, $e^{-f} \in H^1$ and by elliptic regularity, $f \in H^3$. By iteration, f is smooth. □
We can try to replicate the above proof in genus 1. It works easily. In genus 0, you will run into a wall. The crucial point is to know the exact constant in the M-T inequality (for coercivity). That sharp M-T inequality was proved by Moser for the sphere. Try reading up the proof. It is by "symmetrization". Reduces the problem to 1-D. (Not trivial in 1-D though !)

3. The uniformisation theorem for Genus ≥ 2 : The method of sub and super solutions

If $g = e^{-f}g_0$ and we want $K(g) = K$, then $\Delta f = Ke^{-f} - K_0$. It is a semilinear PDE. Let $K = -1$. Let the area of g_0 be normalised to be 1. As before, WLog $K_0 < 0$ everywhere. We shall solve $\Delta f - \epsilon f = -e^{-f} - K_0 - \epsilon f$ for some well chosen $\epsilon > 0$. The point is that $\Delta - \epsilon I$ is invertible by the Fredholm alternative. The strategy is to find f_+, f_- such that $f_+ \geq f_-$, and $\Delta f_+ - \epsilon f_+ \leq -e^{-f_+} - K_0 - \epsilon f_+$ and likewise for f_-. We shall show that such fuctions imply the existence of a solution $f_- \leq f \leq f_+$. By choosing large and small constants, we can trivially find $f_\pm$. Rather than giving the choice of ϵ right away, I shall postpone it to motivate its choice. Consider the sequence f_i satisfying $\Delta f_i - \epsilon f_i = -e^{-f_{i-1}} - K_0 - \epsilon f_{i-1}$ with $f_0 = f_-$. For f_1, note that $\Delta f_1 - \epsilon f_1 \leq \Delta f_- - \epsilon f_-$. Hence $\Delta(f_1 - f_-) \leq \epsilon(f_1 - f_-)$ which means by the min princ. that $f_1 \geq f_-$. If $T(f) = -e^{-f} - \epsilon f - K_0$, note that $T(f)$ is decreasing in f on $[f_-, f_+]$ if $\epsilon > e^{-f_-}$. This is the choice we are after. So $\Delta f_1 - \epsilon f_1 \geq T(f_+) \geq \Delta f_+ - \epsilon f_+$ and hence by the max. princ. $f_1 \leq f_+$. Inductively, we shall prove that $f_- = f_0 \leq f_1 \leq f_2 \leq \ldots \leq f_i \ldots \leq f_+$. Indeed, $\Delta(f_i - f_{i+1}) - \epsilon(f_i - f_{i+1}) = T(f_{i-1}) - T(f_i) \geq 0$. By the max. princ. $f_i \leq f_{i+1}$. $\Delta f_i - \epsilon f_i = T(f_{i-1}) \leq T(f_-) \leq \Delta f_- - \epsilon f_-$ and hence $f_i \geq f_-$. Likewise, $f_i \leq f_+$. The limit $f(x) = \lim_{i\to\infty} f_i(x)$ exists, is measurable, and $f_- \leq f \leq f_+$. Thus it is in L^2 and so is e^{-f}. Let ϕ be a smooth function. Then $\int(\Delta\phi - \epsilon\phi)f_i = \int T(f_{i-1})\phi$. Writing $\Delta\phi = \Delta\phi + C - C$ and $\phi = \phi + C - C$ where $C >> 1$, we see using MCT that f is a distributional solution of the equation. By elliptic regularity and bootstrapping, f is smooth and hence the solution we are looking for.

4. The prescribed Gaussian curvature problem and the Monge-Ampère equation

By computing the first and second fundamental forms, the Gaussian curvature of $(x, y, u(x, y))$ is given by

$$K = \frac{\det(D^2u)}{(1 + \|Du|^2)^2}.$$

More generally, in n-dimensions, it is

$$\det(D^2u) = K(1 + \|Du\|^2)^{(n+2)/2}.$$

The prescribed Gaussian curvature problem asks for a u (possibly with boundary conditions) given a function K. The sign of K plays a huge role in this problem. If you linearise it, and $K > 0$, you get an elliptic equation. (If $K < 0$, you get a wave-type equation.) This is an example of a Monge-Ampère equation $\det(D^2u) = F(u, x, Du)$.

Originally, Monge came up with the equation in the context of envelopes: Just as for a family of curves $F(x, y, p) = 0$ the envelope is $F_p = 0 = F$ (many ways to see this including nearby curves intersect), the envelope of a family of planes $z = px + qy + r(p, q)$ is this along with $x = -r_p, y = -r_q$. Eliminating p, q we get an MA equation. (One more way of looking at this is to notice that the normal at any point is the normal of the plane and hence we have information about the shape operator. This interpretation connects this problem with prescribed Gaussian curvature.) Apparently, this helps also with design of mirrors (mirrors are envelopes of wavefronts). Ampére rigorously analysed the equation. This equation also arises from optimal transportation (again Monge): Consider moving a pile of sand from one location to another with the cost being $T(x, y) = \|x - y\|^2$. So we want to send $x \to U(x)$ (which turns out to be $U(x) = \nabla u$), such that $\int \|x - U(x)\|^2 dx$ is minimum over all U such that local mass conservation holds, i.e., $\det(DU)$ is given (volume element). (This is a Neumann boundary condition.) In the industry, I believe people don't really solve MA but instead using discrete optimal transport algorithms. There is however (as of now theoretical because people don't solve MA for optimal transport in real life) nice "fast" algorithm to solve Neumann OT MA by Berman inspired by Kähler-Einstein metrics!

There is a complex version of this equation which was used by Fefferman to study domains in $\mathbb{C}^n$ and by Yau to prove the Calabi conjecture.

We shall study a "toy" version on a torus: Solve $\det(I + D^2u) = e^{u+f}$ using the method of continuity $\det(I + D^2u_t) = e^{tf+u}$. If there exists a $C^{2,\alpha}$ solution to the original equation, it is unique up to multiplication by a constant and smooth: Firstly, the solution attains a minimum somewhere and hence $I + D^2u > 0$ somewhere. If it every becomes degenerate, we have a contradiction and hence $I + D^2u > 0$ everywhere.