NOTES FOR 2 SEPT (TUESDAY)

1. Recap

(1) Defined connections and proved that they exist on every vector bundle.

2. Connections and curvature

Lemma 2.1. *If two smooth sections $s_1, s_2 : M \to V$ satisfy $s_1 = s_2$ on a neighbourhood U of p, then $\nabla s_1(p) = \nabla s_2(p)$. T(That is, the directional derivative at p depends only on local information about s near p.)*

Proof. Taking $s = s_1 - s_2$, we just have to show that $\nabla s(p) = 0$ if $s = 0$ on U. Indeed, suppose ρ is a bump function, then $\nabla(\rho s) = 0$ because $\rho s = 0$ and $\nabla(\rho s) = d\rho s + \rho \nabla s = 0$ at p. ☐

As a consequence, given a connection on M, and any local section s of V, then $\nabla(\rho s)$ is independent of ρ in a neighbourhood of p (as long as $\rho = 1$ in that neighbourhood). Thus, the Leibniz rule and so applies to local sections too, i.e., given a connection ∇, we have a connection on the vector bundle restricted to an open set.

Locally, suppose $e_1, \ldots, e_r$ is a frame (i.e. a collection of smooth local sections such that every point, they form a basis for the fibre) giving a local trivialisation of V. Then every smooth section is of the form, $s = s^\mu e_\mu$ where s^μ are smooth functions. Therefore,

$$\nabla(s^\mu e_\mu) = ds^\mu \otimes e_\mu + s^\mu \nabla e_\mu = ds^\mu \otimes e_\mu + s^\mu A^\nu_{\ \mu} \otimes e_\nu = (ds^\mu + A^\mu_{\ \nu} s^\nu) \otimes e_\mu \tag{2.1}$$

where $A^\mu_{\ \nu}$ is an $r \times r$ matrix consisting of 1-forms. Note that $\nabla_X s = X(s^\mu)e_\mu + A_{\ \nu}(X)^\mu s^\nu e_\mu$. Suppose we change our trivialisation to $\tilde{e}_1, \tilde{e}_2, \ldots$. Then of course the matrix of 1-forms A will change to $\tilde{A}$. Let us calculate this change. Suppose $\tilde{e}_\mu = g^\nu_{\ \mu} e_\nu$, i.e., $\tilde{e} = eg$ where g is an invertible smooth matrix-valued function. Then since $s = \tilde{s}^\mu \tilde{e}_\mu = s^\nu e_\nu$, we see that $\tilde{e}\vec{\tilde{s}} = eg\vec{\tilde{s}} = e\vec{s}$. Hence $\vec{\tilde{s}} = g^{-1}\vec{s}$. Since $\nabla_X s$ is a section, $\vec{\tilde{\nabla}}_X s = g^{-1}\vec{\nabla}_X s$, i.e., $\vec{\tilde{\nabla}} s = g^{-1}\vec{\nabla} s$. Hence,

$$d\vec{\tilde{s}} + \tilde{A}\vec{\tilde{s}} = g^{-1}(d\vec{s} + A\vec{s}) \Rightarrow d(g^{-1}\vec{s}) + \tilde{A}g^{-1}\vec{s} = g^{-1}(d\vec{s} + A\vec{s})$$

$$-g^{-1}dgg^{-1}\vec{s} + \tilde{A}g^{-1}\vec{s} = g^{-1}A\vec{s} \Rightarrow \tilde{A} = g^{-1}Ag + g^{-1}dg \tag{2.2}$$

In more familiar terms, rewriting $\vec{\tilde{s}} = g\vec{s}$ where g are the transition functions (i.e., replacing g^{-1} by g), we see that $\tilde{A} = gAg^{-1} - dgg^{-1}$.

So A does not change like a tensor. However, the cool thing is that, suppose ∇_1 is one connection. Then, if ∇_2 is any other connection, $(\nabla_2 - \nabla_1)(fs) = f(\nabla_2 - \nabla_1)s$. In other words, the difference of any two connections is an Endomorphism of the vector bundle. Locally, $\tilde{A}_2 - \tilde{A}_1 = g(A_2 - A_1)g^{-1}$. In other words, $A_2 - A_1$ is a section of $End(V) \otimes T^*M$. So the space of connections is an affine space (a vector space without a preferred choice of an origin).

Before going further, we prove a very very useful lemma. (This is like the existence of normal coordinates.)

Lemma 2.2. *Suppose ∇ is a connection on V. Suppose $p \in M$. There exists a trivialisation such that $A(p) = 0$ in this trivialisation.*

Proof. Choose any trivialisation in a neighbourhood U of p. Assume that (x, U) is also a coordinate chart for M such that p corresponds to $x = 0$. Let $\nabla = d + \tilde{A}$ on U. If we change the trivialisation using a transition function g, then $A = g\tilde{A}g^{-1} - dgg^{-1}$. Suppose $\tilde{A}(p) = B_i dx^i$ where B_i are real (or complex) $r \times r$ matrices. Define $g = I + x^i b_i$. For sufficiently small x, g is invertible. Now $g(p) = g^{-1}(p) = I$ and $dg = B_i dx^i = \tilde{A}(p)$. Thus $A(p) = \tilde{A}(p) - \tilde{A}(p) = 0$. □

Note that the trivial bundle $M \times \mathbb{R}^r$ has an obvious connection - the usual directional derivative. Indeed, there is a global trivialisation. Set $A = 0$ and define $\nabla s = d\vec{s}$.

Another point : If $T : V_1 \to V_2$ is a bundle isomorphism, and V_2 has a connection ∇, we can define a connection on V_1 : $(T^*\nabla)s = T^{-1}(\nabla T(s))$. This is called the pullback connection. Locally, the connection matrix of one-forms is $T^{-1}AT + T^{-1}dT$.

So every vector bundle can be equipped with a way to take directional derivatives. There can be more than one way (infinitely many in fact). We can now define the notion of a "constant", rather a "parallel" section.

Definition 2.3. A smooth section s is said to be parallel with respect to a connection ∇ if it satisfies $\nabla s = 0$.

We can do better. Suppose $\gamma : [0, 1] \to M$ is a smooth curve. Assume that s_0 is a vector in $V_{\gamma(0)}$.

Definition 2.4. The parallel transport of s_0 is a section s on a neighbourhood of the image of γ such that $\nabla_{\gamma'(t)} s = 0$ on the image of γ (where we are assuming that $\gamma'(t)$ has been extended arbitrarily to a smooth vector field on a smaller open subset of the neighbourhood on which s is defined).

The neighbourhood does not make any difference in the above definition. The definition locally means this : If we choose a trivialisation and a coordinate chart on the manifold, we are required to solve an ODE : $\frac{d\vec{s}}{dt} + A_{\gamma(t)}(\gamma'(t))\vec{s} = 0$ with $\vec{s}(0) = \vec{s}_0$. Of course this system of ODE has a unique smooth solution for a short period of time. In fact, it can be proven to have a solution for all time.

Now we turn to another notion arising from a connection. What if we want to take the second derivative ? There is a nice way to do this using a connection, but let us return to that later. For now, let us be very naive. Note that ∇ takes sections to vector-valued 1-forms. What if we want to apply ∇ again ? Unfortunately, unless we have a way to differentiate 1-forms, there is no meaning to differentiating $\omega \otimes s$. But we actually do have a way to differentiate 1-forms using the exterior derivative d ! So, define the following map $d^\nabla : \Gamma(V \otimes T^*M) \to \Gamma(V \otimes \Omega^2(M))$ given by $d^\nabla(\omega \otimes s) = d\omega \otimes s + \omega \wedge \nabla s$ and extending it linearly. Of course, $d^\nabla(f\omega \otimes s) = df \wedge \omega \otimes s + f d^\nabla(\omega \otimes s)$. So indeed, tensoriality holds and hence the image of d^∇ is a vector-valued 2-form. Actually, let's take this opportunity to define $d^\nabla : \Gamma(V \otimes \Omega^r M) \to \Gamma(V \otimes \Omega^{r+1} M)$ as $d^\nabla(s \otimes \omega) = \nabla s \wedge \omega + s \otimes d\omega$.

It is natural to ask whether $(d^\nabla)^2 = 0$ on sections (i.e. vector-valued 0-forms). But this is not true ! Indeed, locally, $d^\nabla s = (d\vec{s} + A\vec{s})$. Thus $(d^\nabla)^2 s = d(d\vec{s} + A\vec{s}) + A \wedge (d\vec{s} + A\vec{s}) = 0 + d(A\vec{s}) + A \wedge d\vec{s} + A \wedge A\vec{s} = dA\vec{s} - A \wedge d\vec{s} + A \wedge d\vec{s} + A \wedge A\vec{s} = (dA + A \wedge A)\vec{s} = F\vec{s}$ where F is locally a matrix of 2-forms called the curvature of (V, ∇). In other words, $(d^\nabla)^2 s$ depends linearly on s and not on any derivative of it ! More curiously, if we calculate how F changes when we change the trivialisation, we see that $\tilde{F} = gFg^{-1}$. In other words, F is actually a section of $End(V) \otimes \Omega^2(M)$. (We can do this calculation more invariantly by proving tensoriality, i.e., $(d^\nabla)^2(fs) = f(d^\nabla)^2 s$.)

Definition 2.5. The curvature F of a connection ∇ is a section of $End(V) \otimes \Omega^2(M)$ defined as $Fs = (d^\nabla)^2 s$. It locally has the formula, $F = dA + A \wedge A$.

If V is a line bundle, $A \wedge A = 0$ and $F = dA$ is a global closed 2-form (because $End(L)$ is a trivial bundle).

Here is an interesting observation :

Lemma 2.6. *If* (L, ∇) *is a (real or complex) line bundle, then its curvature* F *is a globally defined closed* 2*-form whose De Rham cohomology class is independent of the connection chosen.*

Proof. We already saw that F is a globally defined close 2-form. Suppose $\nabla_1, \nabla_2 = \nabla_1 + a$ are two connections where a is a section of $End(L) \otimes T^*M$. Noting that $End(L)$ is trivial, a is a globally defined 1-form. Now $F_2 = dA_2 = dA_1 + da = F_1 + da$. Therefore $[F_2] = [F_1]$. □

Real line bundles are actually quite straightforward to study. They are either orientable (and hence trivial) or non-orientable. In either case, $L \otimes L$ has transition functions $g_{\alpha\beta}^2 > 0$. Thus $L \otimes L$ is always a trivial real line bundle.

Complex line bundles are much more complicated and interesting. The De Rham cohomology class $[\frac{\sqrt{-1}}{2\pi}F]$ associated to a complex line bundle L is denoted as $c_1(L)$ and is called the first Chern class of L. (The presence of $\sqrt{-1}$ and 2π is technical. It is done so that whenever you integrate this cohomology class against a 1-dimensional submanifold, you get an integer as the answer.)