NOTES FOR 30 OCT (THURSDAY)

1. Recap

(1) Uniformisation theorem using the variational method. Last step (Euler-Lagrange equations).
(2) Uniformisation using the method of sub and super solutions.
(3) Motivated the Monge-Ampère equation.

2. The prescribed Gaussian curvature problem and the Monge-Ampère equation

We shall study a "toy" version on a torus: Solve $\det(I + D^2u) = e^{u+f}$ using the method of continuity $\det(I + D^2u_t) = e^{tf+u}$. If there exists a $C^{2,\alpha}$ solution to the original equation, it is unique up to multiplication by a constant and smooth: Firstly, the solution attains a minimum somewhere and hence $I + D^2u > 0$ somewhere. If it every becomes degenerate, we have a contradiction and hence $I + D^2u > 0$ everywhere. Now differentiating, $f_i + u_i = tr((I + D^2u)^{-1}D^2u_i)$ and hence u_i satisfies an elliptic equation with $C^{0,\alpha}$ coefficients. By Schauder theory, u_i is $C^{2,\alpha}$ and hence u is $C^{3,\alpha}$. By elliptic bootstrapping, u is smooth. If u_1, u_2 are solutions, then $e^f(e^{u_1} - e^{u_2}) = \det(I + D^2u_1) - \det(I + D^2u_2) = \int_0^1 \det(I + tD^2u_1 + (1 - t)D^2u_2)tr((I + I + tD^2u_1 + (1 - t)D^2u_2)^{-1}D^2(u_1 - u_2)dt$ and hence $u_1 - u_2$ is a constant. Let $S \in [0, 1]$ be the set of all t for which this equation has a smooth solution such that $I + D^2u > 0$. Note that the latter condition is superfluous because of the equation.

(1) S is non-empty: $u = 0$ is the solution at $t = 0$.
(2) S is open: The Banach space implicit function theorem will give us the result if we manage to prove that the linearisation $Lv = \det(I + D^2u)tr((I + D^2u)^{-1}D^2v) - e^{u+f}v$ is an isomorphism. Indeed, $Lv = e^{u+f}(tr((I + D^2u)^{-1}D^2v) - v$. Define $\tilde{L} = Le^{-u-f}$. It is enough to prove that $\tilde{L}$ is an isomorphism from H^{s+2} to H^s for large s (so large that say H^s functions are C^5). Note that $\tilde{L}$ is not formally self-adjoint. However, if you consider $tr(I + sD^2u)^{-1}D^2v) - v$, this is a family of elliptic (and hence Fredholm) operators. The index remains a constant. At $s = 0$, we have a formally self-adjoint operator and hence the index is 0. Thus the kernel of the formal adjoint of $\tilde{L}$ is trivial iff that of $\tilde{L}$ itself is so. The latter is easy to see by the maximum principle.
(3) S is closed: The Arzela-Ascoli theorem and elliptic bootstrapping will give us the result if we prove an *a priori* estimate $\|u_t\|_{C^3} \leq C$. This we shall do below.

2.1. A priori estimates for $\det(I + D^2u) = e^{u+f}$.

(1) C^0 estimate: The maximum principle gives us this easily.
(2) C^2 estimate: We want to use the maximum principle on some cleverly chosen scalar valued function ψ. What function must we choose? Note that if A is positive-definite, then $tr(A) > A$ as matrices. Hence, we can try $\psi = tr(I + D^2u)$ and hope for the best. (Note that if $I + D^2u \leq C$, then by the equation, $I + D^2u > \frac{1}{K}$ for some K too.) Of course at the maximum of ψ, $D^2\psi \leq 0$. We want to choose some matrix-valued function g so that $g^{ij}\psi_{ij}$'s fourth order term can be calculated by differentiating the MA equation itself twice. So we first take ln and differentiate

the equation (let $h^{ij} = (I + D^2u)^{ij}$):

$$h^{ij}u_{pij} = u_p + f_p$$
$$\Rightarrow h^{ij}u_{pqij} - h^{ij}u_{jkq}h^{kl}u_{lip} = u_{pq} + f_{pq}. \tag{2.1}$$

Now

$$\psi_{ij} = \sum_p u_{ijpp} \Rightarrow h^{ij}\psi_{ij} = \sum_p h^{ij}u_{ijpp} = \sum_p (u_{pp} + f_{pp} + h^{ij}u_{jkp}h^{kl}u_{lip}) \geq \Delta f + \Delta u \tag{2.2}$$

At the maximum of ψ, $h^{ij}\psi_{ij} \leq 0$ and hence $\psi \leq C$ at its maximum. Thus $-C \leq D^2u \leq C$. By $W^{2,p}$ estimates, we have a C^2 bound.

(3) C^3 estimate: Again, we want to choose a function W depending on three derivatives of u and apply the maximum principle. It is natural to try $\|D^3u\|^2$ with respect to some norm. Taking cue from ψ above, we can try (as Calabi did)

$$W = h^{ia}h^{jb}h^{kc}u_{ijk}u_{abc}, \tag{2.3}$$

because we already know that $C \geq [h_{ij}] \geq c > 0$ as matrices. Thus $W \leq C$ would imply a C^3 bound on u and we will be done with the proof. As before, we need to differentiate twice (and compare with thrice differentiated MA). It turns out that one gets $W_{ij}h^{ij} \geq C_1W^2 - C_2$ and hence by the maximum principle we are done. This calculation is rather tedious and complicated. (Essentially nothing but completing squares cleverly.) There is a modern way to prove a $C^{2,\alpha}$ bound by Evans-Krylov theory but that requires more PDE tools (like Harnack inequalities).