NOTES FOR 30 SEPT (TUESDAY)

1. Recap

(1) Sobolev spaces on manifolds ("equivalence" of various definitions, including one using tori).

2. Sobolev embedding and compactness

Define $C^{k,\alpha}(M,E)$ as the space of C^k sections of E such that in local coordinates (and frames) they are $C^{k,\alpha}$. The norm on this space is $\|u\|_{C^{k,\alpha}} = \sum_{\mu} \|\vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)}$. This is independent of choices made and is a Banach space. This will be given as a HW problem.

Actually, this is equivalent to the norm $\sum \|\rho_\mu \vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)}$:

Proof. Indeed, firstly, $\sup_x |f(x)g(x)| + \sup_{x,y} \frac{|f(x)g(x)-f(y)g(y)|}{|x-y|^\alpha} \leq \|f\|_{C^{0,\alpha}}\|g\|_{C^{0,\alpha}}$. Hence $\sum \|\rho_\mu \vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)} \leq C\|u\|_{C^{k,\alpha}}$.

Next, if one changes coordinates and trivialisations, the resulting $C^{k,\alpha}$ norms are equivalent (a part of the the HW problem). Therefore, $\|\vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)} \leq \sum_{\nu\neq\mu} \|\rho_\nu \vec{u}_\mu\| + \|\rho_\mu \vec{u}_\mu\|$. Now $\|\rho_\nu \vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)} = \|g_{\nu\mu}\rho_\nu \vec{u}_\nu\|_{C^{k,\alpha}(\bar{U})_\mu} \leq C\|\rho_\nu \vec{u}_\nu\|_{C^{k,\alpha}(\bar{U}_\nu)}$ where the last norm is in the ν coordinates. Hence we are done. $\square$

Firstly, we have the following compactness result :

Lemma 2.1. *Suppose $k \leq l$. If $k < l$ or $0 \leq \beta < \alpha < 1$, then $C^{l,\alpha} \subset C^{k,\beta}$ is a compact embedding.*

Proof. The embedding part is trivial. We shall prove that $C^{0,\alpha} \subset C^0$ is compact (the general case is similar). Let ρ_μ be a partition of unity. If $\|f_n\|_{C^{0,\alpha}} \leq C$, then $\|\rho_\mu f_n\|_{C^{0,\alpha}(\bar{U}_\mu)} \leq C$. By the usual Arzela-Ascoli argument, there is a subsequence (which we shall denote by f_n still) such that $\rho_\mu f_n \to f_\mu$ on $C^0(\bar{U}_\mu)$ for some function $f_\mu : U_\alpha \to \mathbb{R}^r$. (For each μ there is a potentially different subsequence. We choose one for the first μ, then choose a further subsequence for the second μ and so on. There are only finitely many μ.) Clearly f_μ has compact support in U_μ and hence can be extended to a C^0 section of E on M. Now $\|\sum_\mu f_\mu - f_n\|_{C^0(M)} \leq C\sum_\mu \|f_\mu - \rho_\mu f_n\|_{C^0(\bar{U}_\mu)} \to 0$. $\square$

Now we prove Sobolev embedding plus compactness.

Theorem 2.2. *The following inclusions are compact. (Sometimes, this along with the above theorem are referred to as the Sobolev embedding theorems.)*

1. *$H^s(E) \subset H^l(E)$ if $l < s$. (Rellich lemma.)*
2. *$H^s(E) \subset C^a(M,E)$ if $s \geq [\frac{n}{2}] + a + 1$. (Rellich-Kondrachov compactness.)*

Proof. (1) The inclusion part is clear. If f_n is a bounded sequence in $H^s(E)$, then $\rho_\alpha f_n \in H^s(S^1 \times S^1 \ldots)$ is a bounded sequence and by the usual Rellich lemma, it has a convergent subsequence (which abusing notation as usual we still denote by the subscript n) $\rho_\alpha f_n \to f_\alpha$ in $H^l(S^1 \times S^1 \ldots)$. Passing to a further subsequence (that converges a.e.) we see that f_α has support in U_α and hence can be thought of as being a global section on M. By equivalence of norms, $\rho_\alpha f_n \to f_\alpha$ in $H^s(M,E)$. Thus $\sum \rho_\alpha f_n = f_n \to \sum f_\alpha$.

(2) If $f \in H^s(E)$ then $\rho_\alpha f \in H^s(S^1 \times S^1 \ldots)$. Thus $\rho_\alpha f \in C^a(S^1 \times S^1 \ldots)$ by the usual Sobolev embedding on the torus. Hence, $\rho_\alpha f \in C^a(M, E)$ by equivalence of norms. Thus $\sum_\alpha \rho_\alpha f = f \in C^a(M, E)$. Likewise, if $f_n \in H^s(E)$ is bounded, then a subsequence $\rho_\alpha f_n \to f_\alpha$ in $C^a(S^1 \times S^1 \ldots)$. Since f_α is supported on U_α, as before $f_n = \sum \rho_\alpha f_n \to \sum f_\alpha$ in $C^a(M, E)$. □

3. Elliptic operators - Regularity

Now we define the notion of a uniformly elliptic operator : Suppose $(E, h_E, \nabla_E), (F, h_F)$ are smooth bundles with metrics and a metric compatible connection for E on a compact oriented (M, g) where TM is equipped with the Levi-Civita connection. Whenever we use ∇ in what follows, it is made out of ∇_E, ∇_g (Fix h_E, h_F, ∇_E, and g in whatever follows.) First we prove a "structure theorem" for linear PDOs.

Lemma 3.1. *To every linear PDO L of order o with smooth coefficients, there exist smooth maps $a_k : T^*M \otimes T^*M \otimes \ldots T^*M \otimes E \to F$ (where T^*M is repeated k times) such that $L(u) = \sum_{k=0}^{o} a_k \nabla^k u$.*

Proof. We prove this by induction on o. For $o = 0$, by tensoriality, there is such an endomorphism. Assume the result for $0, 1, \ldots, o-1$. Then locally, in a trivialising coordinate chart, $L(u)_\alpha = \sum_{k=0}^{o} a^I_{k,\alpha} \partial_I \vec{u}_\alpha$. If we change the trivialising coordinate chart, then $\vec{u}_\beta = g_{\beta\alpha} \vec{u}_\alpha$, and $\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$ (and the tensor product version of this). The highest order term changes as $a^I_{o,\alpha} \partial_{x,I} \vec{u}_\alpha \to a^I_{o,\alpha} g_{\beta\alpha} \frac{\partial y^J}{\partial x^I} \partial_{y,J} \vec{u}_\beta$, i.e., a_o is a global section of $End(T^*M \otimes T^*M \ldots E, F)$. Hence $L(u) - a_o \nabla^o u$ is a linear PDO of order $o - 1$ and hence by induction we are done. □

The formal adjoint L^*_{form} of L is defined as being a linear PDO of the same order given by $\sum_{k=0}^{o} (\nabla^k)^\dagger \circ a_k^\dagger$. It satisfies (and is equivalent to) $(L^*_{form} u, v) = (u, Lv)$ for smooth u, v.

Definition 3.2. The principal symbol of L is the Endomorphism $\sigma(L) : T^*M \otimes \ldots E \to F$ given by $\sigma(L) = a_o$. A linear PDO L with smooth coefficients is called uniformly elliptic with ellipticity constants $\delta_1, \delta_2 > 0$ if $\delta_1 |v|^2_{h_E(p)} \leq |\sigma_p(L)(\zeta, \zeta, \ldots, \zeta) v|^2_{h_F(p)} \leq \delta_2 |v|^2_{h_E(p)}$ $\forall\ p \in M, |\zeta| = 1, \zeta \in T^*_p M, v \in E_p$ and the principal symbol is invertible. (Please note that δ_1, δ_2 depend on the fixed h_F, h_E obviously.) In particular, the ranks of E and F are required to be the same. (Check that this definition is well-defined.)

It is clear that L is uniformly elliptic (from now on, called "elliptic") if and only if L^*_{form} is so. The ellipticity constants may be chosen to be equal. (Again, the ranks of E and F being the same is important for this.)

Definition 3.3. Suppose f is an L^2 section of F. An L^2 section u is said to be a distributional solution of $Lu = f$ if for every smooth section ϕ of F, $(u, L^*_{form} \phi) = (f, \phi)$. (Please note that we have not defined distributions in general. However, the notion of a distributional solution does not need distributions.)

Next we prove that distributional solutions of elliptic equations are smooth.

Theorem 3.4. *If L is uniformly elliptic and f a smooth section of F. Then if $u \in L^2$ satisfies $Lu = f$ in the sense of distributions then u is smooth. Moreover, if $f \in H^s$, then $u \in H^{s+o}$ and $\|u\|_{H^{s+o}} \leq C_s(\|f\|_{H^s} + \|u\|_{L^2})$ where C_s depends only on h_E, h_F, g, ∇_E, an upper bound on $\|a_k\|_{C^{s+o}}$, and on the ellipticity constants.*

Before proving this theorem, let us prove a "warm up" result.

Theorem 3.5. *If L is uniformly elliptic, u is a smooth section of E, then $\|u\|_{H^{s+o}} \leq C_s(\|Lu\|_{H^s} + \|u\|_{L^2})$.*

The following inequality is useful in the proof of theorem 3.5 and even later on as well.

Lemma 3.6. *If $s^{''} < s^{'} < s$, then for any $f \in H^s(S^1 \times S^1 \ldots, \mathbb{R}^r)$, for any $t > 0$,*

$$\|f\|^2_{s'} \leq \frac{s^{'} - s^{''}}{s - s^{''}} t^{(s-s^{''})/(s^{'}-s^{''})} \|f\|^2_s + \frac{s - s^{'}}{s - s^{''}} t^{-(s-s^{''})/(s-s^{'})} \|f\|^2_{s''} \tag{3.1}$$

Proof. Firstly, we notice the following useful fact of life : If $a, b, t > 0$, $0 < \lambda < 1$, then

$$a^{\lambda} b^{1-\lambda} \leq \lambda t^{1/\lambda} a + (1-\lambda) t^{-1/(1-\lambda)} b \tag{3.2}$$

This fact follows by differentiating the RHS wrt t and finding the absolute minimum. Using this fact,

$$(1+|k|^2)^{s^{'}} \leq \frac{s^{'} - s^{''}}{s - s^{''}} t^{(s-s^{''})/(s^{'}-s^{''})} (1+|k|^2)^s + \frac{s - s^{'}}{s - s^{''}} t^{-(s-s^{''})/(s-s^{'})} (1+|k|^2)^{s^{''}} \tag{3.3}$$

Using this it is easy to see the desired Sobolev space inequality. □

Now we prove that if $u \in H^o$, then $\|u\|_{H^o} \leq C(\|f\|_{L^2} + \|u\|_{L^2})$. Indeed, let $u_n \to u \in H^o$ be smooth sections converging to u (they exist because of a partition-of-unity argument and the corresponding result for the torus). Then, by 3.5, $\|u_n\|_{H^o} \leq C(\|Lu_n\|_{L^2} + \|u_n\|_{L^2})$. Taking limits on both sides, we get the result.

Moreover, given the result for H^o, we can inductively prove it for H^s for all integers $s \geq 0$. For $s = 1$, indeed, since $Lu = f$ a.e. and $f \in H^1$, $\nabla Lu = \nabla f$. Therefore, in the weak sense, $L\nabla u = \nabla f - [\nabla, L]u$. The right hand side is controlled by $C(\|f\|_{H^1} + \|u\|_{L^2})$ by the $s = 0$ result. Therefore, by the $s = 0$ result $\nabla u \in H^o$ and we are done for $s = 1$. Likewise, we can prove it for higher s.