NOTES FOR 4 NOV (TUESDAY)

1. Recap

(1) Solved a toy Monge-Ampère equation on a torus using the method of continuity (the max principle was used for estimates).

2. The Riemann mapping theorem and electrostatics

The Riemann mapping theorem states the following.

Theorem 2.1. *Let $U \subset \mathbb{C}$ be a simply connected open set. There exists an invertible analytic map $f : U \to \mathbb{D}$ (where $\mathbb{D}$ is the unit disc centred at the origin) such that its inverse is also analytic. If there are two such maps, they differ by a Möbius transformation of the disc to itself.*

The uniqueness part follows from standard complex analysis (Schwarz lemma). This part was proven by Poincaré much later. Riemann's original formulation of the existence part of the theorem assumed that the subset U was bounded and had a smooth boundary, i.e., near every boundary point, the boundary could be parametrised as a smooth regular path. We shall prove the theorem under this assumption. The physical idea at play is to pretend that U is an insulator whose boundary is grounded, i.e., has zero electric potential. Then we place a charge at a point $p \in U$. If U is a disc, then choose p to be the centre. In that case, the equipotential curves are concentric circles and the electric field lines radiate outwards. We expect a similar phenonmenon to take place when U is not the unit disc. The corresponding Riemann map is simply the map that takes the equipotential curves to the circles and the field lines to the radial lines. The catch with this "proof" is that we need to rigorously prove that the potential due to a charge with grounded boundary exists. Riemann assumed this fact without proof. (This problem can be phrased as finding the minimum of the electrostatic energy, and it was assumed back in the day that this minimum exists. This fact was termed 'Dirichlet's principle' by Riemann.)

The details of Riemann's proof are as follows.

(1) Existence of a Green's function (potential of a unit charge): There exists a smooth solution u on $\bar{U}$ satisfying $\Delta u = 0$ on U and $u(z) = \ln|z-p|$ on ∂U. This fact is deep and its proof will take some time. (The point is that we have not dealt with elliptic PDE on manifolds with boundary yet.) Consider $G = \ln|z-p| - u$. This function is smooth away from p, $\Delta G = 0$ away from p, $G = 0$ on the boundary, and G corresponds to the potential with a "unit charge" at p. (Rigorously, $\Delta G = \delta(p)$ where δ is the Dirac delta distribution.)

(2) Existence of a harmonic conjugate (a function whose level sets are field lines): There exists a smooth function v on U such that $\nabla v = (-u_y, u_x)$ (that is, its gradient points along the equipotential curves). The function is defined as $v(z) = \int_p^z (-u_y, u_x).\vec{dl}$ along any path $\alpha(t)$ from p to z. The fact that this integral is path-independent follows from simple-connectedness. Indeed, suppose $\alpha(t), \beta(t)$ are two such paths, then consider the piecewise smooth closed path $\gamma(t) : [0,1] \to U$ given by the concatentation of $\alpha(2t)$ and $\beta(2-2t)$. By simple-connectedness, there is a continuous map $H(t,s) : [0,1]\times[0,1] \to U$ such that $H(1,s) = H(0,s) = p \; \forall \, s, H(t,1) =$

$p\ \forall\ t, H(t,0)=\gamma(t)\ \forall\ t$. Actually, it turns out that H can be assumed to be piecewise smooth without loss of generality (Whitney's approximation theorem). Now

$$\int_0^1\int_0^1\left(\frac{\partial(-u_y)}{\partial y}\left(-\frac{\partial y}{\partial t}\frac{\partial x}{\partial s}+\frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right)+\frac{\partial u_x}{\partial x}\left(-\frac{\partial y}{\partial s}\frac{\partial x}{\partial t}+\frac{\partial y}{\partial t}\frac{\partial x}{\partial s}\right)\right)dsdt=0$$

by the harmonicity of u. By the fundamental theorem of calculus, this integral is also equal to $\int\vec{F}.\vec{dl}(s=1)-\int\vec{F}.\vec{dl}(s=0)$ where $\vec{F}=(-u_y,u_x)$. (We have essentially proven Green's theorem in this special case.)

The above argument shows that v is well-defined. It is easy to check that $u+\sqrt{-1}v$ is a holomorphic function (that is, the Cauchy-Riemann equations are met).

(3) Construction of the map: Consider $f(z)=e^{u(z)+\sqrt{-1}v(z)}(z-p)$. This function is holomorphic and takes p to 0. Note that $|f|=|z-p|e^{u(z)}=e^{G(z)}$ and $Arg(f(z))=v+Arg(z-p)$. As expected, the "equipotential" curves are taken to circles and the "field lines" to radial lines. By the mean value property for harmonic functions, no local maximum of G is attained. Proving that f is indeed a biholomorphism taking Ω to $\mathbb{D}$ is more technical.

(4) Proof that the map actually works: On the boundary, i.e., $\partial\Omega\cup\{p\}$, $G\leq 0$. Hence, $G\leq 0$ throughout. This means that $f(\Omega)\subset\overline{\mathbb{D}}$. By the open mapping theorem for holomorphic functions, the image $f(\Omega)$ is open and hence $f(\Omega)\subset\mathbb{D}$. We claim that it is closed too. Suppose $f(z_n)\to b\in\mathbb{D}$. By compactness of $\bar{\Omega}$, a subsequence z_{n_k} (which we shall denote as z_n abusing notation) converges to some $z\in\bar{\Omega}$. If $z_n\in\Omega\to\partial\Omega$, we see that $|f(z_n)|\to 1$. Since $|b|<1$, we see that $z\in\mathbb{D}$ and $f(z)=b$. Thus f is onto. Moreover, $f^{-1}(p)=\{0\}$, and since $f'(p)>0$ we see that the multiplicity of the root at p is 1. Moreover, f does not vanish anywhere else. Note that f is proper and its degree is 1. Thus the pre-image of every regular point (which are dense in the image) is one point. This means that f is actually $1-1$ and $f'\neq 0$ (why?) A holomorphic $1-1$ onto immersion is a biholomorphism.

As mentioned earlier, the solvability of $\Delta u=0$ with Dirichlet boundary conditions is a tricky affair (that Riemann assumed without proof). By means of extending the boundary function to a smooth function and subtraction, we can reduce this problem to solving $\Delta u=f$ with $u=0$ on the boundary. We shall do this now.

3. The Dirichlet problem for the Poisson equation on a smooth domain in $\mathbb{R}^n$ (from Evans' book)

We need to set up Sobolev theory and elliptic theory on compact manifolds-with-boundary and that is a complicated affair. The model space for H^s will not be a torus anymore and instead, near a boundary point it will be $S^1\times\ldots S^1\times[0,\infty)$. On paper one can consider the closure of compactly supported functions in the H^s norm (which is defined using a horizontal Fourier transform) and so on. One can try to come up with a parametrix by solving an ODE in the (horizontal) Fourier space and so on. You can try it as a challenging project (of course all of this is well-known and there in Taylor's books for instance). We shall instead follows Evans' exposition and only deal with smooth domains in $\mathbb{R}^n$.

We define $W^{k,p}(U)$ where $1\leq p\leq\infty$, and $U\subset\mathbb{R}^n$ is open using weak derivatives. Now $W^{k,p}_0(U)$ is the closure of smooth functions with compact support. These are Banach spaces. Using convolution, we can approximate $W^{k,p}(U)$ functions on compact subsets by smooth functions. In fact, using an exhaustion and a partition-of-unity, we can prove that there exist smooth functions on U approximating in $W^{k,p}(U)$. We ideally want to approximate by smooth functions on $\bar{U}$. To this end, assume that $\bar{U}$ is smooth, i.e., its boundary is a smooth manifold of dimension $n-1$. (In fact being C^1 is good enough for this argument.) Using a smooth partition-of-unity we can reduce the case to approximating in a neighbourhood of the boundary (which is given as a graph $x_n = \gamma(x_1, \ldots)$ of a C^1 function locally). Now take a shifted point $x^\epsilon = x + \lambda \epsilon e_n$ for large enough λ, the function $u_\epsilon(x) = u(x^\epsilon)$ can now be convolved to be smooth (by giving ourselves room to mollify). Now we take $\epsilon \to 0$.

3.1. **Assigning boundary values through Trace.** A beautiful idea within PDE theory is to incorporate boundary conditions for a PDE into the definition of the appropriate function space in which the PDE is solved weakly. Indeed, to this end, let us attempt to define the "restriction of a Sobolev function to the boundary". Note that this cannot be done for L^2 functions for instance (they are not well-defined on measure-zero sets). The most naive idea is to approximate by smooth functions in $\bar{U}$ and restrict them to the boundary. However, the result must not depend on the approximation and to this end we need to prove estimates. Here is the formal statement of the trace theorem:
Let $U \subset \mathbb{R}^n$ have a C^1 boundary. Then there exists a bounded linear map $Tr : H^1(U) \to L^2(\partial U)$ such that if u is continuous on $\bar{U}$ and in $H^1(U)$, then $Tr(u) = u|_{\partial U}$. Moreover, the trace of functions in $H^1_0(U)$ is zero. Also, all trace-zero functions are in $H^1_0(U)$

Proof. By means of approximation, wlog u is smooth up to the boundary. We now prove that the restriction linear map is bounded (and hence by continuity, we can extend the trace operator). Using a partition-of-unity and linearity, we can assume wlog that u is supported in a neighbourhood of a boundary point where we can find new coordinates y such that the boundary is $y^n = 0$. By equivalence of norms, we can work in the new coordinates. Now $0 - \int_{y^n=0} u^2 d^{n-1} y = \int \frac{\partial u^2}{\partial y^n} d^n y$ and hence $\|Tr(u)\|^2_{L^2} \leq 2\|u\|_{L^2} \|\nabla u\|_{L^2} \leq C\|u\|^2_{H^1}$. It is easy to see that the approximation is uniform if u is continuous on $\bar{U}$ and hence we are done.
To be cont'd... □