NOTES FOR 4 SEPT (THURSDAY)

1. Recap

(1) Normal trivialisations, parallel transport, and curvature.

2. Connections and curvature

Given connections ∇^v, ∇^w on vector bundles V and W respectively, there exists natural connections on $V \oplus W$, V^* (for this you only need ∇^v), and $V \otimes W$-

(1) $V \oplus W : \nabla^{v \oplus w}(s \oplus t) = \nabla^v s \oplus \nabla^w t$. It is easy to verify that this satisfy all the definition of a connection. Locally, $A^{v\oplus w} = A^v \oplus A^w$ (a block diagonal matrix). Therefore, $F^{v\oplus w} = F^v \oplus F^w$.

(2) $V \otimes W : \nabla^{v\oplus w}(s \otimes t) = \nabla^v s \otimes t + s \otimes \nabla^w t$. Unfortunately, not every section of $V \otimes W$ is of the form $s \otimes t$. It is not obvious that it is even of the form $\sum c_{\alpha\beta} s_\alpha \otimes t_\beta$ where s_α, t_β are global sections.

However, given a p, it is easy to see that there exist global smooth sections s_α, t_β such that $\sum c_{\alpha\beta} s_\alpha \otimes t_\beta = s$ on a neighbourhood U of p. Define $\nabla^{v\oplus w}(\sum c_{\alpha\beta} s_\alpha \otimes t_\beta) = \sum c_{\alpha\beta}(\nabla^v s_\alpha \otimes t_\beta + s_\alpha \otimes \nabla^w t_\beta)$. We have to show that it is well-defined (independent of choices of s_α, t_β) and is genuinely a connection. This will be given as a homework problem.

Locally, $A^{v\otimes w} = A^v \otimes I + I \otimes A^w$ where we are using the Kronecker product of the matrices. Moreover, $F^{v\otimes w} = F^v \otimes I + I \otimes F^w$.

(3) V^* : Define ∇s^* to satisfy $d(s^*(t)) = (\nabla s^*)(t) + s^*(\nabla t)$ where t is any section of V and s^* a section of V^*. This is indeed a connection (easy to see). Locally, suppose e_i is a frame for V, then e^{i*} defined by $e^{i*}(e_j) = \delta^i_j$ is a frame for V^*. In this frame, $(A^*)^j_{\ i} = (\nabla e^{i*})(e_j) = d(e^{i*}(e_j)) - e^{i*}(\nabla e_j) = -A^i_{\ j}$. Thus $A^* = -A^T$. Therefore $F^* = -F^T$.

In the case where $V = L$ is a line bundle, the curvature satisfies $F^* = -F$. Therefore, for a complex line bundle $c_1(L^*) = -c_1(L)$.

(4) If $E = S \oplus Q$, then given a connection ∇ on E, we can define connections on S and Q. Indeed, $\nabla^S s = \pi_1 \circ \nabla s$ where π_1 is the projection to S. So, for example, since $V \otimes V = Alt \oplus Sym$, we see that, given a connection on V, we have a connection on the alternating tensors. (More generally, $V \otimes V \otimes V \ldots = Alt \oplus$ *other things including* $Sym(V)$.) Hence, if we are given a connection on TM, we have a connection on T^*M and hence on $\Omega^k(M)$ for all k.

As a consequence, given a connection on V, we have a naturally defined connection on $V \otimes V \otimes V \ldots$. If $V = L$ is a line bundle equipped with a connection ∇ with curvature F, then $L \otimes L \otimes \ldots$ has a connection whose curvature is kF. So for a complex line bundle, $c_1(L \otimes L \ldots) = kc_1(L)$. In fact, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

Now we specialise further to more important connections.

Definition 2.1. Suppose h is a metric on V. Then a connection ∇ on V is said to be metric compatible with h if for any two sections s_1, s_2, $d(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$.

It turns out that this is equivalent to saying that parallel transport preserves dot products. Locally, choosing an orthonormal frame $e_1, \ldots, e_r$, (i.e. a collection of r smooth local sections such that at every point there are orthonormal) we see that $d(h(e_i, e_j)) = 0 = h(\nabla e_i, e_j) + h(e_i, \nabla e_j) =$

$h(A^k_{\ i}e_k, e_j) + h(e_i, A^k_{\ j}e_k) = A^j_{\ i} + A^i_{\ j}$. Therefore A is a skew-symmetric (skew-Hermitian in the complex case) matrix of 1-forms in a local orthonormal trivialisation. In that trivialisation, $F = dA + A \wedge A$ is a skew-symmetric (or skew-Hermitian in the complex case) matrix of 2-forms.

Note that the trivial connection $\nabla = d$ on a trivial bundle is compatible with the trivial metric.

Theorem 2.2. *On a vector bundle V equipped with a metric h (whether real or complex), there exists a metric compatible connection ∇.*

The proof of this theorem is very similar to the previous one (indeed, replace "trivialisation" with "orthonormal trivialisation" everywhere). Just as before, if we are given one metric compatible connection ∇_0, every other metric compatible connection equals $\nabla_0 + a$ where $a \in \Gamma(End(V) \otimes T^*M)$ is a skew-symmetric (or skew-Hermitian in the complex case) endomorphism-valued 1-form.

Note that suppose we are given a connection ∇^m on T^*M and ∇^v on V, then we have a connection $\nabla^{m\otimes v}$ on $T^*M \otimes V$. Therefore, we can define the second derivative of a section s of V as $\nabla^{m\otimes v}\nabla^v s$. Likewise, we can define higher order derivatives.

Now we define a PDE on a manifold.

Definition 2.3. Suppose V and W be smooth manifolds. Let $C(M,V), C(M,W)$ be the set of smooth maps from M to V, W respectively. A k^{th} order partial differential operator L is a map $L : C(M,V) \to C(M,W)$ such that locally it is of the form $Ls(x) = F(x, s, \partial s, \partial^2 s, \ldots, \partial^k s)$ where F is a smooth function. A PDE is an equation of the form $Lu = f$.
If V and W are vector bundles, then a linear partial differential operator is a map that takes sections of V to sections of W, and satisfies $L(a_1u_1 + a_2u_2) = a_1L(u_1) + a_2L(u_2)$ where a_1, a_2 are constants and $Ls = F(x, s, \ldots)$ locally.

Note that the above notion is well-defined. Indeed, if you change trivialisations and coordinates, you will get a different F but it will remain smooth and depend only on k derivatives of s.
We can finally come up with examples of PDE on manifolds :

(1) Any PDE in $\mathbb{R}^n$ does the job.
(2) More non-trivially, the Laplace equation $\Delta u = f$ on a torus is an example of a second-order linear PDE. A second order non-linear PDE on a torus is $\Delta u = e^u - f$. (If $f > 0$ this PDE turns out to have a unique smooth solution. Note that if $f = 1$, there is an obvious solution, i.e., $u = 0$.)
(3) $Lu = du = f$ where u is a k-form.
(4) $\nabla u = f$ where $u \in \Gamma(V)$ and $f \in \Gamma(V \otimes T^*M)$.
(5) $\nabla^{T^*M} du = f$ where u is a smooth function and f is a $(0,2)$-tensor. This equation is a second order linear PDE.
(6) A harmonic map $f : M \to N$ satisfies a second order nonlinear PDE (that we may write later).
(7) The Ricci flow is a second-order nonlinear PDE for the metric on a manifold.
(8) The Einstein equations in General Relativity are a second-order nonlinear PDE for a Lorentzian metric.
(9) The Navier-Stokes equation is a second-order nonlinear PDE.
(10) The Monge-Ampère equation is a second order nonlinear PDE.

We now come to the a very special metric-compatible connection on TM for a Riemannian manifold (M, g). This connection is determined completely by the metric.

Theorem 2.4. *Suppose (M, g) is a Riemannian manifold. There exists a unique metric compatible connection ∇ on TM such that it is torsion-free, i.e., for any two smooth vector fields X, Y,*

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{2.1}$$

This connection is called the Levi-Civita connection of the metric g. Commonly, its curvature is simply called the curvature of g.

Proof. We will do this in two ways :

(1) *Using coordinates* : Locally, ∇Y has components $(d + A)\vec{Y} = dY^i + A^i_{\ j} Y^j$ where A is an $m \times m$ matrix of 1-forms. So $A^i_{\ j} = \Gamma^i_{jk} dx^k$ where Γ^i_{jk} are a bunch of locally defined functions (the Christoffel symbols). So $\nabla_X Y$ is locally $\frac{\partial Y^i}{\partial x^j} X^j + \Gamma^i_{jk} X^k Y^j$. Take $X = \frac{\partial}{\partial x^a}$ and $Y = \frac{\partial}{\partial y^b}$ (suitably extended to all of M by a bump function). Now the torsion-free property implies that $\nabla_X Y - \nabla_Y X = 0$. In other words, $\Gamma^i_{ab} = \Gamma^i_{ba}$. In any normal coordinate system, at p, metric compatibility means that $A(p)$ is a skew-symmetric matrix, i.e.,

$$\Gamma^i_{ab}(p) = -\Gamma^a_{ib}(p) = -\Gamma^a_{bi}(p) = \Gamma^b_{ai}(p) = \Gamma^b_{ia}(p) = -\Gamma^i_{ba}(p) = -\Gamma^i_{ab}(p) \tag{2.2}$$

which means that $\Gamma^i_{ab}(p) = 0$. So if the LC connection exists, it is unique.

Define the Levi-Civita connection as :$\nabla Y_X(p) = \frac{\partial Y^i}{\partial x^j}(p) X^j(p)$ in any normal coordinate system at p. The fact that this is a connection is easy to see. (Linearly and tensoriality at p are obvious. The Leibniz rule at p is a consequence of the product rule for derivatives.)

(2) *Invariantly* :

$$\begin{aligned}
&g(\nabla_X Y, Z) = g([X, Y] + \nabla_Y X, Z) \\
&= g([X, Y], Z) + Y(X, Z) - g(X, \nabla_Y Z) = g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z] + \nabla_Z Y) \\
&= g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z]) - Z(g(X, Y)) + g(\nabla_Z X, Y) \\
&= g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z]) - Z(g(X, Y)) + g([Z, X], Y) + g(\nabla_X Z, Y) \\
&= g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z]) - Z(g(X, Y)) + g([Z, X], Y) + X(g(Z, Y)) - g(Z, \nabla_X Y)
\end{aligned}$$

$$\Rightarrow 2g(\nabla_X Y, Z) = g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z]) - Z(g(X, Y)) + g([Z, X], Y) + X(g(Z, Y)) \tag{2.3}$$

This determines the connection completely (you can verify that this is indeed a connection) and is called Kozul's formula for the Levi-Civita connection.

□

Using the Kozul formula you can see that the Christoffel symbols have exactly the formula we wrote whilst studying geodesics. In fact, it is not hard to see that a geodesic is simply a curve γ such that $\nabla_{\gamma'}(\gamma') = 0$.

The torsion-free condition appears mysterious but there is a physics way of looking at it involving carrying rods along geodesics which start rotating in the presence of torsion. Indeed, consider the connection ∇ defined on $T\mathbb{R}^3$ as (suppose X, Y, Z are coordinate vector fields - example on mathoverflow),

$$\begin{aligned}
&\nabla_X Y = Z, \nabla_X Y = -Z \\
&\nabla_X Z = -Y, \nabla_Z X = Y \\
&\nabla_Y Z = X, \nabla_Z Y = -X.
\end{aligned} \tag{2.4}$$

A body undergoing parallel translation for this connection spins like an American football: around the axis of motion with speed proportional to its velocity. So the geodesics are straight lines, and this connection preserves the standard metric, but it has torsion and is thus not the Levi-Civita connection.

Actually, there is another way of looking at the torsion-free condition.

Theorem 2.5. *Suppose M is a manifold. Let ∇^* be the induced connection on T^*M from any connection on TM. Then $d^{\nabla^*} : \Omega^1(M) \to \Omega^2(M)$.*

(1) *$(d^{\nabla^*} - d)\omega$ satisfies tensoriality and hence there exists a tensor $T \in \Gamma(T^{**}M \simeq TM \otimes \Omega^2(M))$ such that $T_\omega(\_,\_) = (d^{\nabla^*} - d)(\omega)$.*

(2) *$T(X,Y) = \omega(\nabla_X Y - \nabla_Y X - [X,Y])$. Thus for the Levi-Civita connection, $d^{\nabla^*} = d$.*

Proof. (1) $(d^{\nabla^*} - d)(f\omega) = df \wedge \omega + f d^{\nabla^*}\omega - df \wedge \omega - f d\omega = f(d^{\nabla^*} - d)\omega$. Hence, by tensoriality there exists such a tensor T (T is called the Torsion tensor of ∇).

(2) To be continued...

$\square$