Lecture 1

Vamsi Pritham Pingali

IISc

Logistics

Logistics

1. Webpage :
   http://math.iisc.ac.in/~vamsipingali/teaching/

Logistics

1. Webpage : `http://math.iisc.ac.in/~vamsipingali/teaching/`
2. HW - 25%, Midterm - 25%, Final/Presentation - 50 % (The HW will be put up on the webpage)

Logistics

1. Webpage : http://math.iisc.ac.in/~vamsipingali/teaching/
2. HW - 25%, Midterm - 25%, Final/Presentation - 50 % (The HW will be put up on the webpage)
3. Office - N -23.

Logistics

1. Webpage :
   http://math.iisc.ac.in/~vamsipingali/teaching/
2. HW - 25%, Midterm - 25%, Final/Presentation - 50 % (The HW will be put up on the webpage)
3. Office - N -23.
4. Prereqs - A first course on manifolds, some analysis (Fourier analysis and a little bit of function spaces and measure theory), multivariable calculus, and functional analysis (up to and including the spectral theorem for compact self-adjoint operators). The functional analysis part can be read from the appendix in Evans' book.

What is geometric analysis?

What is geometric analysis?

1. Geometric analysis is the usage of PDE (

What is geometric analysis?

1. Geometric analysis is the usage of PDE (particularly nonlinear PDE) to study geometric, topological, complex analytic, and algebro-geometric problems.

What is geometric analysis?

1. Geometric analysis is the usage of PDE (particularly nonlinear PDE) to study geometric, topological, complex analytic, and algebro-geometric problems.
2. Linear PDE have been used (

What is geometric analysis?

1. Geometric analysis is the usage of PDE (particularly nonlinear PDE) to study geometric, topological, complex analytic, and algebro-geometric problems.
2. Linear PDE have been used (and continue to be used, for instance by the Demailly school of algebraic and analytic geometry)

What is geometric analysis?

1. Geometric analysis is the usage of PDE (particularly nonlinear PDE) to study geometric, topological, complex analytic, and algebro-geometric problems.
2. Linear PDE have been used (and continue to be used, for instance by the Demailly school of algebraic and analytic geometry) since a long time (

What is geometric analysis?

1. Geometric analysis is the usage of PDE (particularly nonlinear PDE) to study geometric, topological, complex analytic, and algebro-geometric problems.
2. Linear PDE have been used (and continue to be used, for instance by the Demailly school of algebraic and analytic geometry) since a long time (Hodge theory, Atiyah-Singer index theorem, etc)

What is geometric analysis?

1. Geometric analysis is the usage of PDE (particularly nonlinear PDE) to study geometric, topological, complex analytic, and algebro-geometric problems.
2. Linear PDE have been used (and continue to be used, for instance by the Demailly school of algebraic and analytic geometry) since a long time (Hodge theory, Atiyah-Singer index theorem, etc)
3. But nonlinear PDE gained prominence in the last 5 decades.

Examples of success stories (Yau, Uhlenbeck, Taubes, Donaldson, Perelman, Hamilton, etc)

Examples of success stories (Yau, Uhlenbeck, Taubes, Donaldson, Perelman, Hamilton, etc)

1. The Poincaré conjecture using the Ricci flow.

Examples of success stories (Yau, Uhlenbeck, Taubes, Donaldson, Perelman, Hamilton, etc)

1. The Poincaré conjecture using the Ricci flow.
2. The Yamabe problem and the related positive mass theorem.

Examples of success stories (Yau, Uhlenbeck, Taubes, Donaldson, Perelman, Hamilton, etc)

1. The Poincaré conjecture using the Ricci flow.
2. The Yamabe problem and the related positive mass theorem.
3. Frankel's conjecture.

Examples of success stories (Yau, Uhlenbeck, Taubes, Donaldson, Perelman, Hamilton, etc)

1. The Poincaré conjecture using the Ricci flow.
2. The Yamabe problem and the related positive mass theorem.
3. Frankel's conjecture.
4. Minimal surfaces.

Examples of success stories (Yau, Uhlenbeck, Taubes, Donaldson, Perelman, Hamilton, etc)

1. The Poincaré conjecture using the Ricci flow.
2. The Yamabe problem and the related positive mass theorem.
3. Frankel's conjecture.
4. Minimal surfaces.
5. Calabi conjecture and Kähler-Einstein metrics.

Examples of success stories (Yau, Uhlenbeck, Taubes, Donaldson, Perelman, Hamilton, etc)

1. The Poincaré conjecture using the Ricci flow.
2. The Yamabe problem and the related positive mass theorem.
3. Frankel's conjecture.
4. Minimal surfaces.
5. Calabi conjecture and Kähler-Einstein metrics.
6. Gauge theory and topology.

Examples of success stories (Yau, Uhlenbeck, Taubes, Donaldson, Perelman, Hamilton, etc)

1. The Poincaré conjecture using the Ricci flow.
2. The Yamabe problem and the related positive mass theorem.
3. Frankel's conjecture.
4. Minimal surfaces.
5. Calabi conjecture and Kähler-Einstein metrics.
6. Gauge theory and topology.
7. The deformation invariance of plurigenera.

Goals of the course (famous last words)

Goals of the course (famous last words)

1. How to write PDE on manifolds.

Goals of the course (famous last words)

1. How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (

Goals of the course (famous last words)

1. How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (What is a PDE on $\mathbb{R}^n$? It is an equation of the form $F(u, \nabla u, D^2 u, \ldots) = 0$. If you want to write it on a manifold, what is $D^2 u$?

Goals of the course (famous last words)

1. How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (What is a PDE on $\mathbb{R}^n$? It is an equation of the form $F(u, \nabla u, D^2 u, \ldots) = 0$. If you want to write it on a manifold, what is $D^2 u$? If you use coordinates to define it, then changing coordinates gives a different PDE.

Goals of the course (famous last words)

1. How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (What is a PDE on $\mathbb{R}^n$? It is an equation of the form $F(u, \nabla u, D^2 u, \ldots) = 0$. If you want to write it on a manifold, what is $D^2 u$? If you use coordinates to define it, then changing coordinates gives a different PDE. So we need some additional structure to even write PDE on manifolds.

Goals of the course (famous last words)

1. How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (What is a PDE on $\mathbb{R}^n$? It is an equation of the form $F(u, \nabla u, D^2 u, \ldots) = 0$. If you want to write it on a manifold, what is $D^2 u$? If you use coordinates to define it, then changing coordinates gives a different PDE. So we need some additional structure to even write PDE on manifolds. The structure we will use is a Riemannian metric.)

Goals of the course (famous last words)

1. How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (What is a PDE on $\mathbb{R}^n$? It is an equation of the form $F(u, \nabla u, D^2 u, \ldots) = 0$. If you want to write it on a manifold, what is $D^2 u$? If you use coordinates to define it, then changing coordinates gives a different PDE. So we need some additional structure to even write PDE on manifolds. The structure we will use is a Riemannian metric.)
2. How to prove existence and uniqueness of solutions for linear elliptic (and hopefully some nonlinear) PDE.

Goals of the course (famous last words)

1. How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (What is a PDE on $\mathbb{R}^n$? It is an equation of the form $F(u, \nabla u, D^2 u, \ldots) = 0$. If you want to write it on a manifold, what is $D^2 u$? If you use coordinates to define it, then changing coordinates gives a different PDE. So we need some additional structure to even write PDE on manifolds. The structure we will use is a Riemannian metric.)
2. How to prove existence and uniqueness of solutions for linear elliptic (and hopefully some nonlinear) PDE.
3. Hopefully a little bit about parabolic PDE as well.

Goals of the course (famous last words)

1. How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (What is a PDE on $\mathbb{R}^n$? It is an equation of the form $F(u, \nabla u, D^2u, \ldots) = 0$. If you want to write it on a manifold, what is D^2u? If you use coordinates to define it, then changing coordinates gives a different PDE. So we need some additional structure to even write PDE on manifolds. The structure we will use is a Riemannian metric.)
2. How to prove existence and uniqueness of solutions for linear elliptic (and hopefully some nonlinear) PDE.
3. Hopefully a little bit about parabolic PDE as well.
4. Why you should care about studying PDE on manifolds. (

Goals of the course (famous last words)

1. How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (What is a PDE on $\mathbb{R}^n$? It is an equation of the form $F(u, \nabla u, D^2 u, \ldots) = 0$. If you want to write it on a manifold, what is $D^2 u$? If you use coordinates to define it, then changing coordinates gives a different PDE. So we need some additional structure to even write PDE on manifolds. The structure we will use is a Riemannian metric.)
2. How to prove existence and uniqueness of solutions for linear elliptic (and hopefully some nonlinear) PDE.
3. Hopefully a little bit about parabolic PDE as well.
4. Why you should care about studying PDE on manifolds. (Hopefully, some Hodge theory and the uniformisation theorem.)

The Poisson ODE

The Poisson ODE

- Suppose we want to solve the ODE $u^{''} = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function.

The Poisson ODE

- Suppose we want to solve the ODE $u^{''} = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function.
- $u^{'}(x) = u^{'}(0) + \int_0^x f(t)dt$ and $u(x) = u(x + 2\pi)$ implies that $u^{'}(x) = u^{'}(x + 2\pi)$ (

The Poisson ODE

- Suppose we want to solve the ODE $u^{''} = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function.
- $u^{'}(x) = u^{'}(0) + \int_0^x f(t)dt$ and $u(x) = u(x+2\pi)$ implies that $u^{'}(x) = u^{'}(x+2\pi)$ (in fact they are equivalent if $u(0) = u(2\pi)$).

The Poisson ODE

- Suppose we want to solve the ODE $u^{''} = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function.
- $u^{'}(x) = u^{'}(0) + \int_0^x f(t)dt$ and $u(x) = u(x + 2\pi)$ implies that $u^{'}(x) = u^{'}(x + 2\pi)$ (in fact they are equivalent if $u(0) = u(2\pi)$). Thus $\int_0^{2\pi} f(t)dt = 0$.

The Poisson ODE

- Suppose we want to solve the ODE $u^{''} = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function.
- $u^{'}(x) = u^{'}(0) + \int_0^x f(t)dt$ and $u(x) = u(x+2\pi)$ implies that $u^{'}(x) = u^{'}(x+2\pi)$ (in fact they are equivalent if $u(0) = u(2\pi)$). Thus $\int_0^{2\pi} f(t)dt = 0$.
- This is a necessary and sufficient (by the periodicity of f, $\int_x^{x+2\pi} f(t)dt = \int_0^{2\pi} f(t)dt$) condition. (Smoothness is guaranteed by the fundamental theorem of calculus.)

The Poisson ODE

- Suppose we want to solve the ODE $u^{''} = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function.
- $u^{'}(x) = u^{'}(0) + \int_0^x f(t)dt$ and $u(x) = u(x + 2\pi)$ implies that $u^{'}(x) = u^{'}(x + 2\pi)$ (in fact they are equivalent if $u(0) = u(2\pi)$). Thus $\int_0^{2\pi} f(t)dt = 0$.
- This is a necessary and sufficient (by the periodicity of f, $\int_x^{x+2\pi} f(t)dt = \int_0^{2\pi} f(t)dt$) condition. (Smoothness is guaranteed by the fundamental theorem of calculus.)
- In other words,

The Poisson ODE

- Suppose we want to solve the ODE $u^{''} = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function.
- $u^{'}(x) = u^{'}(0) + \int_0^x f(t)dt$ and $u(x) = u(x+2\pi)$ implies that $u^{'}(x) = u^{'}(x+2\pi)$ (in fact they are equivalent if $u(0) = u(2\pi)$). Thus $\int_0^{2\pi} f(t)dt = 0$.
- This is a necessary and sufficient (by the periodicity of f, $\int_x^{x+2\pi} f(t)dt = \int_0^{2\pi} f(t)dt$) condition. (Smoothness is guaranteed by the fundamental theorem of calculus.)
- In other words, there is a unique-upto-a-constant smooth periodic solution of the ODE if and only if

The Poisson ODE

- Suppose we want to solve the ODE $u^{''} = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function.
- $u^{'}(x) = u^{'}(0) + \int_0^x f(t)dt$ and $u(x) = u(x + 2\pi)$ implies that $u^{'}(x) = u^{'}(x + 2\pi)$ (in fact they are equivalent if $u(0) = u(2\pi)$). Thus $\int_0^{2\pi} f(t)dt = 0$.
- This is a necessary and sufficient (by the periodicity of f, $\int_x^{x+2\pi} f(t)dt = \int_0^{2\pi} f(t)dt$) condition. (Smoothness is guaranteed by the fundamental theorem of calculus.)
- In other words, there is a unique-upto-a-constant smooth periodic solution of the ODE if and only if f is smooth, periodic, and satisfies $\int_0^{2\pi} f(t)dt = 0$.

The Poisson ODE

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^{∞},

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^∞, the map $T : C^\infty \to C^\infty$ given by $T(u) = u^{''}$ has kernel precisely the constants.

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^{∞}, the map $T : C^{\infty} \to C^{\infty}$ given by $T(u) = u''$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v \rangle = \int_0^{2\pi} uvdx$,

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^{∞}, the map $T : C^{\infty} \to C^{\infty}$ given by $T(u) = u^{''}$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v \rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^{∞}, the map $T : C^{\infty} \to C^{\infty}$ given by $T(u) = u^{''}$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v \rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$.

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^{∞}, the map $T : C^{\infty} \to C^{\infty}$ given by $T(u) = u^{''}$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v \rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$.
- This is very similar to finite-dimensional linear algebra.

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^∞, the map $T : C^\infty \to C^\infty$ given by $T(u) = u''$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v\rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$.
- This is very similar to finite-dimensional linear algebra. Moreover, by the fundamental theorem of calculus,

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^{∞}, the map $T : C^{\infty} \to C^{\infty}$ given by $T(u) = u^{''}$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v \rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$.
- This is very similar to finite-dimensional linear algebra. Moreover, by the fundamental theorem of calculus, if f is k-times continuously differentiable (will be denoted as C^k from now on), then u is C^{k+2}.

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^{∞}, the map $T : C^{\infty} \to C^{\infty}$ given by $T(u) = u^{''}$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v \rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$.
- This is very similar to finite-dimensional linear algebra. Moreover, by the fundamental theorem of calculus, if f is k-times continuously differentiable (will be denoted as C^k from now on), then u is C^{k+2}.
- These observations are not coincidences!

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^∞, the map $T : C^\infty \to C^\infty$ given by $T(u) = u''$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v \rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$.
- This is very similar to finite-dimensional linear algebra. Moreover, by the fundamental theorem of calculus, if f is k-times continuously differentiable (will be denoted as C^k from now on), then u is C^{k+2}.
- These observations are not coincidences!
- Later on, we will see that many PDE

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^∞, the map $T : C^\infty \to C^\infty$ given by $T(u) = u''$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v\rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$.
- This is very similar to finite-dimensional linear algebra. Moreover, by the fundamental theorem of calculus, if f is k-times continuously differentiable (will be denoted as C^k from now on), then u is C^{k+2}.
- These observations are not coincidences!
- Later on, we will see that many PDE (the so-called elliptic PDE) satisfy similar properties.

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^∞, the map $T : C^\infty \to C^\infty$ given by $T(u) = u''$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v \rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$.
- This is very similar to finite-dimensional linear algebra. Moreover, by the fundamental theorem of calculus, if f is k-times continuously differentiable (will be denoted as C^k from now on), then u is C^{k+2}.
- These observations are not coincidences!
- Later on, we will see that many PDE (the so-called elliptic PDE) satisfy similar properties. However, to prove such things, we cannot rely on a direct formula for the solution unlike the case of ODE.

The Poisson ODE

- Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^{∞}, the map $T : C^{\infty} \to C^{\infty}$ given by $T(u) = u''$ has kernel precisely the constants.
- Moreover, equipping this vector space with the inner product $\langle u, v \rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$.
- This is very similar to finite-dimensional linear algebra. Moreover, by the fundamental theorem of calculus, if f is k-times continuously differentiable (will be denoted as C^k from now on), then u is C^{k+2}.
- These observations are not coincidences!
- Later on, we will see that many PDE (the so-called elliptic PDE) satisfy similar properties. However, to prove such things, we cannot rely on a direct formula for the solution unlike the case of ODE. So we need a more abstract, theoretical method.

Fourier analysis

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k) e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx$ and likewise for f.

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k)e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and likewise for f.
- We see that $\hat{u}(k)k^2 = -\hat{f}(k)$.

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k)e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and likewise for f.
- We see that $\hat{u}(k)k^2 = -\hat{f}(k)$. In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$.

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k)e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and likewise for f.
- We see that $\hat{u}(k)k^2 = -\hat{f}(k)$. In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$.
- In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant.

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k)e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and likewise for f.
- We see that $\hat{u}(k)k^2 = -\hat{f}(k)$. In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$.
- In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant.
- Moreover, since as sharp changes in music

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k)e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and likewise for f.
- We see that $\hat{u}(k)k^2 = -\hat{f}(k)$. In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$.
- In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant.
- Moreover, since as sharp changes in music (think of opera music) correspond to very shrill sounds,

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k) e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx$ and likewise for f.
- We see that $\hat{u}(k)k^2 = -\hat{f}(k)$. In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx$.
- In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant.
- Moreover, since as sharp changes in music (think of opera music) correspond to very shrill sounds, if the high-frequency Fourier components are "small",

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k) e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx$ and likewise for f.
- We see that $\hat{u}(k)k^2 = -\hat{f}(k)$. In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$.
- In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant.
- Moreover, since as sharp changes in music (think of opera music) correspond to very shrill sounds, if the high-frequency Fourier components are "small", then the function is very "smooth" (

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k)e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and likewise for f.
- We see that $\hat{u}(k)k^2 = -\hat{f}(k)$. In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$.
- In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant.
- Moreover, since as sharp changes in music (think of opera music) correspond to very shrill sounds, if the high-frequency Fourier components are "small", then the function is very "smooth" (melodious notes are not too shrill).

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k)e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and likewise for f.
- We see that $\hat{u}(k)k^2 = -\hat{f}(k)$. In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$.
- In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant.
- Moreover, since as sharp changes in music (think of opera music) correspond to very shrill sounds, if the high-frequency Fourier components are "small", then the function is very "smooth" (melodious notes are not too shrill).
- Since $\hat{u}(k) = \frac{\hat{f}(k)}{k^2}$, u behaves more smoothly than f does.

Fourier analysis

- Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k) e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx$ and likewise for f.
- We see that $\hat{u}(k) k^2 = -\hat{f}(k)$. In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$.
- In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant.
- Moreover, since as sharp changes in music (think of opera music) correspond to very shrill sounds, if the high-frequency Fourier components are "small", then the function is very "smooth" (melodious notes are not too shrill).
- Since $\hat{u}(k) = \frac{\hat{f}(k)}{k^2}$, u behaves more smoothly than f does. So if f is a smooth function, we expect u to be so as well.

Fourier analysis

Fourier analysis

- make things rigorous, firstly, notice that the Fourier coefficients make sense for any integrable function.

- make things rigorous, firstly, notice that the Fourier coefficients make sense for any integrable function. The convergence of Fourier series is a subtle phenomenon though.

- make things rigorous, firstly, notice that the Fourier coefficients make sense for any integrable function. The convergence of Fourier series is a subtle phenomenon though.
- For example, there exist continuous functions whose Fourier series do not convergence pointwise at some points.

Fourier analysis

- make things rigorous, firstly, notice that the Fourier coefficients make sense for any integrable function. The convergence of Fourier series is a subtle phenomenon though.
- For example, there exist continuous functions whose Fourier series do not convergence pointwise at some points.
- Already this is beginning to hint that expecting results like

Fourier analysis

- make things rigorous, firstly, notice that the Fourier coefficients make sense for any integrable function. The convergence of Fourier series is a subtle phenomenon though.
- For example, there exist continuous functions whose Fourier series do not convergence pointwise at some points.
- Already this is beginning to hint that expecting results like "If f is C^k, then u is C^{k+2}" is a bad idea from the Fourier-analytic point of view.

Fourier analysis

- make things rigorous, firstly, notice that the Fourier coefficients make sense for any integrable function. The convergence of Fourier series is a subtle phenomenon though.
- For example, there exist continuous functions whose Fourier series do not convergence pointwise at some points.
- Already this is beginning to hint that expecting results like "If f is C^k, then u is C^{k+2}" is a bad idea from the Fourier-analytic point of view. In fact for PDE, this expectation is false.

Fourier analysis

- make things rigorous, firstly, notice that the Fourier coefficients make sense for any integrable function. The convergence of Fourier series is a subtle phenomenon though.
- For example, there exist continuous functions whose Fourier series do not convergence pointwise at some points.
- Already this is beginning to hint that expecting results like "If f is C^k, then u is C^{k+2}" is a bad idea from the Fourier-analytic point of view. In fact for PDE, this expectation is false.
-

Fourier analysis - Useful results

Fourier analysis - Useful results

- Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if

- Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if its Fourier series converges in the L^2 norm to it.

Fourier analysis - Useful results

- Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if its Fourier series converges in the L^2 norm to it. Moreover, if a_k is in l^2, then $\sum a_k e^{ikx}$ converges in L^2.

Fourier analysis - Useful results

- Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if its Fourier series converges in the L^2 norm to it. Moreover, if a_k is in l^2, then $\sum a_k e^{ikx}$ converges in L^2.
- Parseval-Plancherel : The Fourier series transform is an isometric isomorphism between $L^2([0, 2\pi])$ and l^2.

Fourier analysis - Useful results

- Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if its Fourier series converges in the L^2 norm to it. Moreover, if a_k is in l^2, then $\sum a_k e^{ikx}$ converges in L^2.
- Parseval-Plancherel : The Fourier series transform is an isometric isomorphism between $L^2([0, 2\pi])$ and l^2.
- Let $C^{0,\alpha}$ $(0 < \alpha < 1)$ consist of all Hölder continuous

Fourier analysis - Useful results

- Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if its Fourier series converges in the L^2 norm to it. Moreover, if a_k is in l^2, then $\sum a_k e^{ikx}$ converges in L^2.
- Parseval-Plancherel : The Fourier series transform is an isometric isomorphism between $L^2([0, 2\pi])$ and l^2.
- Let $C^{0,\alpha}$ $(0 < \alpha < 1)$ consist of all Hölder continuous 2π-periodic functions g, i.e., periodic functions g such that $|g(x) - g(y)| \leq C|x - y|^\alpha$ for all x, y.

Fourier analysis - Useful results

- Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if its Fourier series converges in the L^2 norm to it. Moreover, if a_k is in l^2, then $\sum a_k e^{ikx}$ converges in L^2.
- Parseval-Plancherel : The Fourier series transform is an isometric isomorphism between $L^2([0, 2\pi])$ and l^2.
- Let $C^{0,\alpha}$ $(0 < \alpha < 1)$ consist of all Hölder continuous 2π-periodic functions g, i.e., periodic functions g such that $|g(x) - g(y)| \leq C|x - y|^\alpha$ for all x, y. Note that if f is in C^1, then f is Hölder continuous.

Fourier analysis - Useful results

- Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if its Fourier series converges in the L^2 norm to it. Moreover, if a_k is in l^2, then $\sum a_k e^{ikx}$ converges in L^2.
- Parseval-Plancherel : The Fourier series transform is an isometric isomorphism between $L^2([0, 2\pi])$ and l^2.
- Let $C^{0,\alpha}$ $(0 < \alpha < 1)$ consist of all Hölder continuous 2π-periodic functions g, i.e., periodic functions g such that $|g(x) - g(y)| \leq C|x - y|^\alpha$ for all x, y. Note that if f is in C^1, then f is Hölder continuous.
  Theorem : If $f \in C^{0,\alpha}$ then $|\hat{f}(k)| \leq \frac{K}{|k|^\alpha}$ $\forall$ $|k| \geq 1$.

Proof

$$2\pi\frac{\widehat{f(x+h)}(k)-\hat{f}(k)}{h^\alpha}=\int_0^{2\pi}\frac{f(x+h)-f(x)}{h^\alpha}e^{-ikx}dx$$

$$\Rightarrow\left|\frac{\widehat{f(x+h)}(k)-\hat{f}(k)}{h^\alpha}\right|\leq C \qquad (1)$$

$$\left|\int_0^{2\pi}\frac{f(x+h)-f(x)}{h^\alpha}e^{-ikx}dx\right|=\left|\int_h^{2\pi+h}\frac{f(y)}{h^\alpha}e^{-ik(y-h)}\right.$$

$$\left.-\int_0^{2\pi}\frac{f(x)}{h^\alpha}e^{-ikx}dx\right|$$

$$=\left|-\int_0^{h}\frac{f(y)}{h^\alpha}e^{-ik(y-h)}dy+\int_{2\pi}^{2\pi+h}\frac{f(y)}{h^\alpha}e^{-ik(y-h)}dy\right.$$

$$\left.+\frac{1}{h^\alpha}\int_0^{2\pi}e^{-ikx}f(x)(e^{ikh}-1)dx\right|$$

$$=\left|\frac{1}{h^\alpha}\int_0^{2\pi}e^{-ikx}f(x)(e^{ikh}-1)dx\right|=|\hat{f}(k)|\frac{|e^{ikh}-1|}{h^\alpha} \qquad (2)$$

Fourier analysis - useful results continued...

Take $h = \frac{1}{k}$. Using 1 and 2 we see that $|\hat{f}(k)| \leq \frac{K}{k^{\alpha}}$. $\square$

Take $h = \frac{1}{k}$. Using 1 and 2 we see that $|\hat{f}(k)| \leq \frac{K}{k^{\alpha}}$. □

- If $f \in C^{0,\alpha}$, the Fourier series converges uniformly to f.

Fourier analysis - useful results continued...

Take $h = \frac{1}{k}$. Using 1 and 2 we see that $|\hat{f}(k)| \leq \frac{K}{k^{\alpha}}$. $\square$

- If $f \in C^{0,\alpha}$, the Fourier series converges uniformly to f.
- If $f \in C^1$ then $\hat{f'} = ik\hat{f}(k)$.

Fourier analysis - useful results continued...

Take $h = \frac{1}{k}$. Using 1 and 2 we see that $|\hat{f}(k)| \leq \frac{K}{k^{\alpha}}$. $\square$

- If $f \in C^{0,\alpha}$, the Fourier series converges uniformly to f.
- If $f \in C^1$ then $\hat{f'} = ik\hat{f}(k)$. This holds for higher derivatives too.

Fourier analysis - useful results continued...

Take $h = \frac{1}{k}$. Using 1 and 2 we see that $|\hat{f}(k)| \leq \frac{K}{k^{\alpha}}$. □

- If $f \in C^{0,\alpha}$, the Fourier series converges uniformly to f.
- If $f \in C^1$ then $\hat{f'} = ik\hat{f}(k)$. This holds for higher derivatives too.

Proof.

$$\hat{f'} = \frac{1}{2\pi}\int_0^{2\pi} f'(x)e^{-ikx}dx = -\frac{1}{2\pi}\int_0^{2\pi} f(x)(e^{-ikx})'dx$$
$$= \frac{1}{2\pi}\int_0^{2\pi} ikf(x)e^{-ikx}dx = ik\hat{f}(k) \quad (3)$$

□

Fourier analysis - useful results

Fourier analysis - useful results

- If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial).

Fourier analysis - useful results

- If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial). Also the Fourier series of f and its derivatives converge uniformly.

Fourier analysis - useful results

- If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial). Also the Fourier series of f and its derivatives converge uniformly. Conversely, if a_k are rapidly decaying, then they are the Fourier coefficients of a smooth function (with convergence being uniform).

Proof.

If f is smooth, then $\widehat{f^{(l)}}(k) = (ik)^l \hat{f}(k)$.

Fourier analysis - useful results

- If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial). Also the Fourier series of f and its derivatives converge uniformly. Conversely, if a_k are rapidly decaying, then they are the Fourier coefficients of a smooth function (with convergence being uniform).

Proof.

If f is smooth, then $\hat{f^{(l)}}(k) = (ik)^l \hat{f}(k)$. Since $\hat{f^{(l)}}(k)$ is bounded, $\hat{f}(k)$ is rapidly decaying.

Fourier analysis - useful results

- If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial). Also the Fourier series of f and its derivatives converge uniformly. Conversely, if a_k are rapidly decaying, then they are the Fourier coefficients of a smooth function (with convergence being uniform).

Proof.

If f is smooth, then $\widehat{f^{(l)}}(k) = (ik)^l \hat{f}(k)$. Since $\widehat{f^{(l)}}(k)$ is bounded, $\hat{f}(k)$ is rapidly decaying. By one of the earlier theorems, the convergence is uniform.
If $|a_k| \leq C_l |k|^{-l}$, then by the Weierstrass M-test (choosing $l > 1$),

Fourier analysis - useful results

- If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial). Also the Fourier series of f and its derivatives converge uniformly. Conversely, if a_k are rapidly decaying, then they are the Fourier coefficients of a smooth function (with convergence being uniform).

Proof.

If f is smooth, then $\hat{f^{(l)}}(k) = (ik)^l \hat{f}(k)$. Since $\hat{f^{(l)}}(k)$ is bounded, $\hat{f}(k)$ is rapidly decaying. By one of the earlier theorems, the convergence is uniform.
If $|a_k| \leq C_l |k|^{-l}$, then by the Weierstrass M-test (choosing $l > 1$), we see that $\sum a_k e^{ikx}$ converges uniformly to a continuous function u.

Fourier analysis - useful results

- If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial). Also the Fourier series of f and its derivatives converge uniformly. Conversely, if a_k are rapidly decaying, then they are the Fourier coefficients of a smooth function (with convergence being uniform).

Proof.

If f is smooth, then $\widehat{f^{(l)}}(k) = (ik)^l \hat{f}(k)$. Since $\widehat{f^{(l)}}(k)$ is bounded, $\hat{f}(k)$ is rapidly decaying. By one of the earlier theorems, the convergence is uniform.

If $|a_k| \leq C_l |k|^{-l}$, then by the Weierstrass M-test (choosing $l > 1$), we see that $\sum a_k e^{ikx}$ converges uniformly to a continuous function u. The same argument also shows that $\sum (ik)^l a_k e^{ikx}$ converges uniformly to u_l.

Fourier analysis - useful results

- If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial). Also the Fourier series of f and its derivatives converge uniformly. Conversely, if a_k are rapidly decaying, then they are the Fourier coefficients of a smooth function (with convergence being uniform).

Proof.

If f is smooth, then $\hat{f^{(l)}}(k) = (ik)^l \hat{f}(k)$. Since $\hat{f^{(l)}}(k)$ is bounded, $\hat{f}(k)$ is rapidly decaying. By one of the earlier theorems, the convergence is uniform.
If $|a_k| \leq C_l |k|^{-l}$, then by the Weierstrass M-test (choosing $l > 1$), we see that $\sum a_k e^{ikx}$ converges uniformly to a continuous function u. The same argument also shows that $\sum (ik)^l a_k e^{ikx}$ converges uniformly to u_l. It is easy to see (fundamental theorem of calculus and interchange of summation and integration) that $u_l = u^{(l)}$. □

Thank you