# NOTES FOR 6 NOV (THURSDAY)

## 1. Recap

(1) Riemann mapping via electrostatics.
(2) Definition of Sobolev spaces and density of smooth functions on bounded smooth domains.
(3) Trace theorem - proved the existence of the trace operator.

## 2. The Dirichlet problem for the Poisson equation on a smooth domain in $\mathbb{R}^n$ (from Evans' book)

2.1. **Assigning boundary values through Trace.** Let $U \subset \mathbb{R}^n$ have a $C^1$ boundary. Then there exists a bounded linear map $Tr : H^1(U) \to L^2(\partial U)$ such that if $u$ is continuous on $\bar{U}$ and in $H^1(U)$, then $Tr(u) = u|_{\partial U}$. Moreover, the trace of functions in $H^1_0(U)$ is zero. Also, all trace-zero functions are in $H^1_0(U)$

*Proof.* Cont'd....
Now we prove that every trace-zero function can be approximated by compactly supported smooth ones in $H^1$. Firstly, we can assume using a partition-of-unity and equivalence of norms that $u$ is compactly supported on the upper-half space (and replace $U$ with the upper-half space). The same trick used for global smooth approximation ('shifting the point') works to approximate $u$ in $H^1$ on the upper-half space by smooth ones $u_m$ on the closed upper-half space (having compact support in the closed upper-half space). Now their restrictions to the boundary converge in $L^2(\partial U)$ to 0. We now take an exhaustion of the upper-half space as follows: let $0 \leq \zeta \leq 1$ be a smooth function that is 1 on $[0, 1]$ and 0 outside $[0, 2]$. Write $w_m = u(1 - \zeta_m)$ where $\zeta_k = \zeta(kx^n)$. We claim that $w_m$ (which is smooth and compactly supported on the open upper half-space) converges to $u$ in $H^1$. Now

$$\int \|\nabla w_m - \nabla u\|^2 d^n x \leq C \int |\zeta_m|^2 \|\nabla u\|^2 d^n x + Cm^2 \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u|^2 d^{n-1} x dt$$
$$=: A + B \tag{2.1}$$

Note that $A \to 0$ as $m \to \infty$ because of DCT. As for the second term, we need to use the fundamental theorem of calculus:

$$|u_m(x', x^n)| \leq |u_m(x', 0)| + \int_0^{x^n} \|\nabla u_m\| dt$$

(2.2)

$$\Rightarrow \int |u_m(x', x^n)|^2 d^{n-1} x' \leq C(\int |u_m(x', x^n)|^2 d^{n-1} x' |u_m(x', 0)|^2 d^{n-1} x' + x_n \int_0^{x^n} \int \|\nabla u_m(x', t)\|^2 d^{n-1} x' dt.$$

As $m \to \infty$ we see that

$$\int |u(x', x^n)|^2 d^{n-1} x' \leq C x^n \int_0^{x^n} \int \|\nabla u(x', t)\|^2 d^{n-1} x' dt. \tag{2.3}$$

Substituting into the inequality we derived earlier, we see using Cauchy-Schwarz that as $m \to \infty$ the term $B \to 0$. Indeed,

$$Cm^2 \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u|^2 d^{n-1}x dt \leq Cm^2 \int_0^{2/m} x^n \int_0^{x^n} \int \|\nabla u(x', t)\|^2 d^{n-1}x' dt dx^n$$
$$\leq Cm^2 \int_0^{2/m} x^n \int_0^{2/m} \int \|\nabla u(x', t)\|^2 d^{n-1}x' dt dx^n$$
$$\leq C \int_0^{2/m} \int \|\nabla u(x', t)\|^2 d^{n-1}x' dt \to 0 \tag{2.4}$$

by DCT. □

2.2. **Sobolev embedding.** We shall first prove the Gagliardo-Sobolev-Nirenberg inequality: Let $1 \leq p < n$ and $u$ be smooth and compactly supported in $\mathbb{R}^n$. We want to prove an inequality of the type $\|u\|_{L^q} \leq C\|\nabla u\|_{L^p}$. If indeed such an inequality holds, suppose we change the scale, that is, consider $u_t(x) = u(tx)$. Then $\int |u_t|^q = \frac{1}{t^n} \int |u|^q$ and $\int \|\nabla u_t\|^p = \frac{t^p}{t^n} \int \|\nabla u\|^p$. Thus $q = \frac{np}{n-p}$ if we want a scale-invariant inequality (otherwise if we zoom in or out, we will get a contradiction). Indeed, this inequality is true for $C^1$ compactly supported functions as we shall now show:
As is usual in maths, starting with the simplest non-trivial case $p = 1$ is best. In fact, if we manage to prove this case, apply the estimate to $v = |u|^\gamma$ where $\gamma > 1$ is cleverly chosen as follows to get the result for higher $p$ (why is $v$ $C^1$?):

$$\left( \int |u|^{\gamma n/(n-1)} \right)^{(n-1)/n} \leq C \int \|\nabla |u|^\gamma\| = C\gamma \int |u|^{\gamma - 1} \|\nabla u\|$$
$$\leq C\gamma (\int |u|^{(\gamma-1)p/(p-1)})^{(p-1)/p} \|\nabla u\|_{L^p}. \tag{2.5}$$

So choosing $\gamma = \frac{p(n-1)}{n-p} > 1$ works.
Now we prove that case of $p = 1$. The idea is to use FTC and Hölder repeatedly.

$$|u(x)| \leq \int_{-\infty}^{\infty} \|\nabla u\|(x^1, \ldots, y^i, \ldots) dy^i \ \forall\ 1 \leq i \leq n$$
$$\Rightarrow |u(x)|^{n/(n-1)} \leq \Pi_{i=1}^n \left( \int_{-\infty}^{\infty} \|\nabla u\|(x^1, \ldots, y^i, \ldots) dy^i \right)^{1/(n-1)}. \tag{2.6}$$

Now integrate w.r.t $x^1$ on both sides. One term is independent of $x^1$ in the product and pulls out.

$$\int |u(x)|^{n/(n-1)} dx^1 \leq \int_{-\infty}^{\infty} \|\nabla u\|(x^1, \ldots, y^i, \ldots) dy^1 \int \left( \int_{-\infty}^{\infty} \|\nabla u\|(x^1, \ldots, y^i, \ldots) dy^i \right)^{1/(n-1)} dx^1. \tag{2.7}$$

At this juncture, we use the general Hölder inequality to get

$$\int |u(x)|^{n/(n-1)} dx^1 \leq \int_{-\infty}^{\infty} \|\nabla u\|(x^1, \ldots, y^i, \ldots) dy^1 \Pi_{i=2}^n \left( \int \int_{-\infty}^{\infty} \|\nabla u\|(x^1, \ldots, y^i, \ldots) dx^1 dy^i \right)^{1/(n-1)}. \tag{2.8}$$

Now integrate w.r.t $x^2$ on both sides. Again one term is independent of $x^2$ and pulls out. The integral of the product of the other terms is again estimated using the general Hölder inequality to result in the following.

$$\int |u(x)|^{n/(n-1)}dx^1dx^2 \leq \left(\int \int_{-\infty}^{\infty} \|\nabla u\|(x^1,\ldots,y^i,\ldots)dx^1dy^2\right)^{1/(n-1)} \left(\int_{-\infty}^{\infty} \|\nabla u\|(x^1,\ldots,y^i,\ldots)dx^2dy^1\right)^{1/(n-1)}$$

$$\times\Pi_{i=3}^{n}\left(\int \int \int_{-\infty}^{\infty} \|\nabla u\|(x^1,\ldots,y^i,\ldots)dx^1dx^2dy^i\right)^{1/(n-1)}. \tag{2.9}$$

Continuing like this, we get the desired estimate. □

To generalise this inequality to Sobolev embedding for $U$, we need a device to extend a given function from $u$ to a compactly supported function in $\mathbb{R}^n$ with controlled Sobolev norm. This is the topic of the next theorem.

**Theorem 2.1.** *Let $1 \leq p < \infty$. (The theorem holds even for $p = \infty$ but we won't bother.) Suppose $U \subset \mathbb{R}^n$ is a bounded open subset with $C^1$ boundary. Select any bounded open set $V$ such that $U \subset\subset V$. There exists a bounded linear operator (an 'extension map') $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that for each $u \in W^{1,p}(U)$, $Eu = u$ a.e. in $U$, $Eu$ has support in $V$ and $\|Eu\|_{W^{1,p}} \leq C\|u\|_{W^{1,p}(U)}$.*

*Proof.* By approximation, assume that $u$ is smooth up to the boundary. (We need $p < \infty$ here.) First assume that $\partial U$ is flat near $x^0$ and lies in $x^n = 0$. Choose some open ball $B$ centred at the origin such that $B^+ \subset \bar{U}$. Now define $v(x) = u(x)$ when $x^n \geq 0$ and $x$ is in the ball, and $v(x) = -3u(x', -x^n) + 4u(x', -x^n/2)$ when $x^n < 0$ and $x$ is in the ball. It is easy to check that $v$ is in $C^1(\bar{B})$ and satisfies $\|v\|_{W^{1,p}(B)} \leq C\|v\|_{W^{1,p}(B^+)}$. Now use the implicit function theorem, equivalence of norms under diffeomorphisms, and a partition-of-unity (with supports such that there are all contained in $V$) to patch up these local extensions. □

Now we prove that if $U \subset \mathbb{R}^n$ is bounded open with $C^1$ boundary, $1 \leq p < n$ and $u \in W^{1,p}(U)$, then $u \in L^q(U)$ with the estimate $\|u\|_{L^q}(U) \leq C\|u\|_{W^{1,p}(U)}$:
Indeed, extend $u$ to $v = Eu$ having compact support in $\mathbb{R}^n$, and with controlled $W^{1,p}$ norm. Now by convolution, there exist smooth compactly supported $v_m$ converging to $v$ in $W^{1,p}(\mathbb{R}^n)$. The GSN inequality now implies the result. □
Remark: For $u \in W_0^{1,p}(U)$, we can do better. The RHS can be taken to be $\|\nabla u\|_{L^p(U)}$. This is also called Poincaré's inequality.

Now we look at what happens when $n < p < \infty$. We have the Morrey inequality.

**Theorem 2.2.** *For all $u \in C^1(\mathbb{R}^n)$, $\|u\|_{C^{0,\gamma}} \leq C\|u\|_{W^{1,p}}$ where $\gamma = 1 - \frac{n}{p}$.*

*Proof.* First we prove a $C^0$ bound and then upgrade it to Hölder. Naively, we would use FTC and try to bound the derivative using $W^{1,p}$. But the latter involves an $n$-dimensional integral (as opposed to 1-dimensional). So we might have better luck bounding an average. Motivated by this observation, consider $|u(x)| \leq \fint_{B(x,1)} |u(x) - u(y)|dy + \fint_{B(x,1)} |u(y)|dy$. The last term is $\leq C\|u\|_{L^p(\mathbb{R}^n)}$. We need to bound the first term. Let $w \in \partial B(0,1)$ and $0 < s < r$.

$$|u(x+sw)-u(x)| \leq \int_0^s \|\nabla u\|(x+tw)dt$$
$$\int_{\partial B(0,1)} |u(x+sw)-u(x)|dA \leq \int_0^s \int_{\partial B(0,1)} \|\nabla u\|(x+tw)dAdt$$
$$\leq \int_{B(0,s)} \frac{\|\nabla u\|(z)}{\|z-x\|^{n-1}}dz$$
$$\Rightarrow \int_{\partial B(x,s)} \|u(z)-u(x)\|dA(z) \leq s^{n-1} \int_{B(0,r)} \frac{\|\nabla u\|(z)}{\|z-x\|^{n-1}}dz$$
$$\Rightarrow \int_{B(x,r)} \|u(y)-u(x)\|dy \leq \frac{r^n}{n} \int_{B(x,r)} \int_{B(0,r)} \frac{\|\nabla u\|(z)}{\|z-x\|^{n-1}}dz. \tag{2.10}$$

By Hölder's inequality, we get a $C^0$ bound. Cont'd..... □