NOTES FOR 7 OCT (TUESDAY)

1. Recap

(1) Sobolev embedding and compactness on manifolds.
(2) Elliptic operators -definition, statement of elliptic regularity, and proof of the interpolation inequality (in preparation for the proof of elliptic estimates for smooth solutions).

2. Elliptic regularity

Now we prove the aforementioned theorem (for smooth solutions as opposed to weak solutions). The full elliptic regularity result for weak L^2 solutions is more complicated (and can be found in Folland's book or in Kodaira's book). We shall give a sketch of the ideas later.

Proof. First cover the manifold by coordinate trivialisations (which are balls of radius 2) V_μ such that balls of radius 1/2 U_μ (note that $\bar{U}_\mu \subset V_\mu$) continue to cover. Now choose a fixed partition-of-unity $\rho_{0,\mu}$ having compact supports in U_μ. We choose the square of size 1 centred at the origin and identify the edges to form tori. We can now identify any function with compact support inside this square to be a function on the torus. From now onwards, whenever we choose a smaller cover, it will be a refinement of this cover, and the identification with the tori is *fixed*. But each time we refine the cover, the partition-of-unity will also change to have support in the new cover. So we must be careful.
Writing $u = \sum \rho_{0,\mu} u$, we see that if we can prove that $\|\rho_{0,\mu}u\|_{H^{s+o}} \leq C_s(\|L(\rho_{0,\mu}u)\|_{H^s} + \|\rho_{0,\mu}u\|_{L^2})$ we will be done. Indeed (from now onwards all constants depending on s (and on the ellipticity constants and upper bounds on the coefficients) will be denoted by abuse of notation as C_s),

$$\|u\|_{H^{s+o}} \leq C_s \sum_\mu (\|L(\rho_{0,\mu}u)\|_{H^s} + \|\rho_{0,\mu}u\|_{L^2}) \leq \tilde{C}_s \sum_\mu (\|\rho_{0,\mu}Lu\|_{H^s} + \|u\|_{L^2}) + C_s \sum_\mu \|[L,\rho_{0,\mu}]u\|_{H^s}$$
$$\leq \tilde{C}_s(\|Lu\|_{H^s} + \|u\|_{L^2}) + C_s\|u\|_{H^{s+o-1}} \tag{2.1}$$

Using the interpolation inequality we see that $C_s\|u\|_{H^{s+o-1}} \leq \frac{1}{2}\|u\|_{H^{s+o}} + C\|u\|_{L^2}$. Thus we have reduced the problem to proving $\|\rho_{0,\mu}u\|_{H^{s+o}} \leq C_s(\|L(\rho_{0,\mu}u)\|_{H^s} + \|\rho_{0,\mu}u\|_{L^2})$.

Cover the square up by a fine enough cover $\tilde{U}_\nu$ of balls with centres c_ν such that the balls of half the radius continue to cover and $L_\nu = \tilde{\rho}_\mu L + (1-\tilde{\rho}_\mu)a_o(c)^I\partial_I$ is elliptic on $\tilde{U}_\nu$ for any $c \in \tilde{U}_\nu$ with uniform ellipticity constants. Now choose a partition of unity ψ_ν (with supports in balls of half the radius of $\tilde{U}_\nu$) and consider $\tilde{L} = \sum_\nu \psi_\nu L_\nu$. Note that $\tilde{L}$ is elliptic and $\tilde{L}u = Lu$ when u has support in U_μ. Thus, we have reduced the problem to proving the estimate on a flat torus with the trivial vector bundle (but with variable coefficients).

The rough idea is to cover the torus with lots of open sets such that the operator is not far from a constant coefficient one on those sets, i.e., $L - L(p_\mu)$ is small. Then we know that $\|\rho_\mu u\|_{H^{s+o}} \leq C_s(\|L(p_\mu)\rho_\mu u\|_{H^s} + \|\rho_\mu u\|_{L^2}) \leq C_s(\|L\rho_\mu u\|_{H^s} + \|\rho_\mu u\|_{L^2}) + C_s\|(L - L(p,\mu))\rho_\mu u\|_{H^s}$. Note that a single C_s works independent of what p_μ are simply because C_s depends only on the ellipticity constants and upper bounds on the coefficients. If the last term is smaller than $\frac{1}{2}\|u\|_{H^{s+o}}$ (for instance), then we are done. But if we make the cover small, we risk making the other factor large. This is the problem.

Firstly, we claim that it is enough to prove the estimate for $s=0$. Indeed, if this is done, then

$$\begin{aligned}\|\partial_i u\|_{H^o} &\leq C(\|L(\partial_i u)\|_{L^2}+\|\partial_i u\|_{L^2}) \leq C(\partial_i(Lu)\|_{L^2}+\|[L,\partial_i]u\|_{L^2}+\|u\|_{H^o}) \\ &\Rightarrow \|u\|_{H^{o+1}} \leq C(\|Lu\|_{H^1}+\|u|_{L^2})\end{aligned} \tag{2.2}$$

Inductively we can prove this for a general s.

Suppose we choose a fine enough cover of the torus so that $\|(L-L(p_\mu))u\|_{L^2} \leq \frac{1}{n^{2^{n\theta 2^n}} 2^{2^n} C_0}\|u\|_{H^0}$. Then of course $\|(\rho_\mu(L-L(p_\mu))u\|_{L^2} \leq \frac{1}{n^{2^{n\theta 2^n}} 2^{2^n} C_0}\|u\|_{H^o}$ because $\rho_\mu \leq 1$. Fix such a cover and a partition-of-unity (we will not make it any finer than this). Therefore,

$$\begin{aligned}\frac{1}{2}\|u\|_{H^o} &\leq \sum_\mu C_0(\|L\rho_\mu u\|_{L^2}+\|\rho_\mu u\|_{L^2}) + C_0\sum_\mu \|[(L-L(p,\mu)),\rho_\mu]u\|_{H^s} \\ &\leq C_0\sum_\mu(\|L\rho_\mu u\|_{L^2}+\|\rho_\mu u\|_{L^2}) + C_1\|u\|_{H^{o-1}}\end{aligned}$$

$$\leq C_0(\|Lu\|_{L^2}+\|u\|_{L^2}) + C_0\sum_\mu \|[L,\rho_\mu]u\|_{L^2} + C_1\|u\|_{H^{o-1}} \leq C_0(\|Lu\|_{L^2}+\|u\|_{L^2}) + C_2\|u\|_{H^{o-1}} \tag{2.3}$$

If we can prove that $\|u\|_{H^{o-1}} \leq \frac{1}{3C_2}\|u\|_{H^o}+C\|u\|_{L^2}$, we will be done. Indeed, this follows from the interpolation inequality for Sobolev spaces. $\square$