NOTES FOR 9 OCT (THURSDAY)

1. Recap

(1) Proved elliptic estimates for smooth solutions. Did so by first reducing to a variable coefficient operator on a torus (by cleverly extending an elliptic operator from an open subset of $\mathbb{R}^n$ to a larger open subset and making it constant coefficient near the boundary of a large square). Then chose a fine enough cover so that the operator is approximately constant coefficient and used the result for constant coefficient operators.

2. Elliptic regularity

Now we return to the full elliptic regularity result. The key ideas are

(1) Distributions : The space of distributions of order s $H^{-s}(M,E)$ is defined to be the metric space completion of L^2 under the norm $\|v\|_{H^{-s}} = \|F_v\| = \sup_{u\in H^s} \frac{|(u,v)_{L^2}|}{\|u\|_{H^s}}$. Note that if $v \in L^2$, then $\|v\|_{-s} \leq \|v\|_{L^2}$. Also, if $v \in H^{-s}$, then considering equivalence classes of Cauchy sequences, we get a linear functional on H^s, namely, F_v such that $|F_v(u)| \leq \|v\|_{-s}\|u\|_s$. It is not hard to prove using functional analysis that $(H^s)^* \simeq H^{-s}$ and that the map is an isometry (and hence H^{-s} is a Hilbert space). Indeed, the map T defined by $v \to F_v$ is a linear map from H^{-s} to $(H^s)^*$. Now if $v_n \in L^2$, then $\|F_{v_n}\| = \|v_n\|_{H^{-s}}$ by definition. If $v_n \to v$ in H^{-s}, then $F_{v_n} \to F_v$ and $\|F_v\| = \|v\|_{H^{-s}}$. We just have to prove that T is onto. Since $Im(T)$ is closed, we need to prove that the orthogonal complement is 0. Suppose $L \in Imt(T)^{\perp}$. Then letting L denote the Riesz representative in H^s, $Tv(L) = 0$ for all v. That is, $F_v(L) = 0$ for all v. In particular, this is true for $v \in L^2$ and hence $(v, L)_{L^2} = 0$ for all $v \in L^2$ and hence $L = 0$.
Some easy functional analysis also allows one to conclude that given $G \in (H^{-s})^*$, there is a unique $u \in H^s$ such that $G(v) = (u,v)_{L^2}$ (Exercise). As in the case of a torus, we define the derivative of a distribution through "integration-by-parts". The Sobolev inclusion and Rellich compactness still hold. (Exercise)
In fact, we claim that $v \in H^{-s}$ if and only if $\rho_\mu^2 v \in H^{-s}(S^1 \times S^1 \ldots, \mathbb{R}^r)$ (where $\rho_\mu^2 v(u) = v(\rho_\mu^2 u)$) where U_μ is a trivialising coordinate cover and ρ_μ^2 is a partition-of-unity). Moreover, the H^{-s} norm is equivalent to $\sum_\mu \|\rho_\mu^2 v\|_{H^{-s}(S^1\times S^1\ldots)}$. Furthermore, one can prove that if $u \in L^2$ is a distributional solution of $Lu = f$ where $f \in H^{-o}$, then an elliptic estimate holds. This will be part of a HW.

(2) Difference quotients : Let u be a vector-valued H^{-s} distribution on the torus and $0 < h \leq 1$. The difference quotient $\Delta_{h,e_i}u$ is a vector-valued H^{-s} distribution defined as $\Delta_{h,e_i}u(v) = u(\Delta_{-h}v) = u(\frac{v(x-he_i)-v(x)}{h})$ for all $v \in H^s$. Here is a beautiful result about difference quotients.

Theorem 2.1. *Let $s \in \mathbb{R}$. The following spaces are on the torus.*
(a) *If $|u\|_{H^{s+1}} \leq C$, then $\|\Delta_h u\|_{H^s} \leq C \ \forall\ 0 < h < 1$.*
(b) *Conversely, if $u \in H^s$ and $\|\Delta_h u\|_{H^s} \leq C \ \forall\ 0 < h < 1,\ e_i$, then $|u\|^2_{H^{s+1}} \leq nC^2 + \|u\|^2_{H^s}$.*

(c) *If* $|\alpha| = l$ *and* $u \in H^{s+l}$, *then* $\|[a_\alpha(x)\partial^\alpha, \Delta_h]u\|_{H^s} \leq C\|u\|_{H^{s+l}}$ $\forall\ 0 < h < 1$ *where* C *depends only on the upper bounds on the* C^{s+1} *norms of the coefficient* a_α.

Proof. The Fourier coefficients of the distribution $\Delta_h u$ can be easily calculated to be $\hat{\Delta_h u}(k) = \frac{\hat{u}(k)e^{\sqrt{-1}k_i h} - \hat{u}(k)}{h}$.

(a)

$$\|\hat{\Delta_h u}\|^2_{H^s} = \sum_k |\hat{u}(k)|^2(1+|k|^2)^s k_i^2 \frac{\sin^2(\frac{k_i h}{2})}{(\frac{k_i h}{2})^2} \leq C^2 \tag{2.1}$$

(b) For each i, take $h \to 0$ and use Fatou's lemma to conclude that

$$C \geq \sum_k |\hat{u}(k)|^2(1+|k|^2)^s k_i^2 \tag{2.2}$$

We add over i to get the result.

(c) Let smooth functions $u_n \to u$ in H^{s+l},

$$\|[a_\alpha(x)\partial^\alpha, \Delta_h]u_n\|_{H^s} = \|\Delta_h a_\alpha \partial^\alpha u_n(x+h)\|_{H^s} \leq C\|u_n(x)\|_{H^s} \tag{2.3}$$

Taking $n \to \infty$ we get the result.

□

We can combine the above ideas to prove the desired theorem. (HW) A very brief sketch: Indeed, if $f \in L^2$ and $u \in L^2$ such that $Lu = f$ in the sense of distributions, then first we reduced proving that $u \in H^l$ to the torus case with constant coefficients using a partition-of-unity. Then, $L\Delta_h u = \Delta_h f$. Hence, $\|\Delta_h u\|_{L^2} \leq C_{-l}(\|\Delta_h f\|_{H^{-l}} + \|\Delta_h u\|_{H^{-l}}) \leq C(\|f\|_{L^2} + \|u\|_{L^2})$. Thus, $u \in H^1$. Now, $L\nabla u = \nabla f - [L, \nabla]u$ in the sense of distributions. Since $\nabla u \in L^2$ and the right-hand side is controlled in H^{-l}, the previous reasoning shows that $\nabla u \in H^1$ and so on. Once we prove that $u \in H^l$, the desired inequality follows by approximation with smooth sections and using the theorem above.

3. Elliptic operators-Fredholmness

We shall prove that

Theorem 3.1. *If* $L : H^o \to L^2$ *is elliptic, then*

1. *$Im(L) \subset L^2$ is closed, and the kernel and cokernel are finite-dimensional.*
2. *The kernel consists of smooth functions.*
3. *The Cokernel $\simeq ker(L^*) : L^2 \to (H^o)^*$ consists of smooth functions and $Coker(L) \simeq ker(L^*_{form})$.*

Proof. (1) This will follow from the construction of parametrices.
Firstly, we have the following lemma :

Lemma 3.2. *If U_μ is a coordinate trivialising open cover of M, ρ_μ^2 is a partition-of-unity subordinate to it, and $K_\mu : H^s(S^1 \times S^1 \ldots, \mathbb{R}^r) \to H^s(S^1 \times \ldots, \mathbb{R}^r)$ are compact operators, then $K : H^s(M, E) \to H^s(M, E)$ given by $K(u) = \sum_\mu \rho_\mu K_\mu(\rho_\mu u)$ is also compact where we secretly extend functions supported on U_μ to $S^1 \times S^1 \ldots$ and conversely, functions on $S^1 \times S^1 \ldots$ having support in the image of U_μ are extended by 0 to the manifold.*

Proof. Indeed, if u_n is bounded sequence in H^s, then $\rho_\mu u_n$ is bounded in $H^s(S^1 \times \ldots)$ and hence there is a convergent subsequence of $K_\mu(\rho_\mu u_n)$ (depending on μ which we will as usual denote by the subscript n shamelessly) converging to u_μ. Since $\rho_\mu u_\mu$ has compact support, it can be extended to all of M and the convergence happens in the Sobolev norm on M. Thus $K(u_n) = \sum_\mu \rho_\mu K_\mu(\rho_\mu u_n) \to \sum_\mu \rho_\mu u_\mu$ in H^s. Hence K is compact. □

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