HW 1 (due on 19th August (Tuesday) in the class)

Please write your answers clearly and rigorously. Write your name in plain lettering (as opposed to cursive) and also staple all the pages.

1. Prove that $C^{k,\alpha}(\bar{U})$ is a Banach space, where $U \subset \mathbb{R}^n$ is a bounded open set whose boundary is a smooth submanifold of $\mathbb{R}^n$. (Recall that $\|u\|_{C^{k,\alpha}(\bar{U})} := \max_{\bar{U}} \|u\| + \ldots + \max_{\bar{U}} \|D^k u\| + \sum_{\|I\|=k} \sup_{x \neq y \in \bar{U}} \frac{\|D^I u(x) - D^I u(y)\|}{\|x-y\|^{\alpha}}$.)

2. Let $\beta < \alpha$. Prove that a bounded sequence $u_n \in C^{k,\alpha}(\bar{U})$ has a convergent subsequence in $C^{k,\beta}(\bar{U})$. Show that if $\beta = \alpha$ this conclusion is false by exhibiting a counterexample.

3. (Problem 17, chapter 5 in Evans) Assume that $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose U is bounded and $u \in W^{1,p}(U)$ for some $1 \leq p < \infty$. Show that $v = F(u)$ is in $W^{1,p}(U)$ and $v_{x_i} = F'(u) u_{x_i}$.

4. Show that if $u, v \in H^s(S^1 \times S^1 \ldots)$ where $s > \frac{n}{2}$, then $uv \in H^s(S^1 \times S^1 \ldots)$ and $\|uv\|_{H^s} \leq C \|u\|_{H^s} \|v\|_{H^s}$.