HW 6 (due on Oct 30)

Please write your answers clearly and rigorously. Write your name in plain lettering (as opposed to cursive) and also staple all the pages.

1. Prove the Moser-Trudinger inequality $\int e^{\beta u^2} \leq C$ if $\|u\|_{H^1} \leq 1$ on a compact oriented surface.

2. Let $\int u = 0$ and $u \in H^1$. Let e_n be the eigenfunctions of the Laplacian with eigenvalues λ_n. Then $u = \sum_n u_n e_n$ in L^2. Prove that

   (a) $u_0 = 0$.

   (b) The eigenvalues of Δ are non-positive. (Denote the largest non-zero one by λ_1.)

   (c) $\|\nabla u\|_{L^2}^2 = -\sum_n |u_n|^2 \lambda_n$

   (d) Deduce the Poincaré inequality.

3. Let f_1, f_2 be smooth positive functions and $p \geq 1$ be any real number. Solve

$$\Delta u = f_1 |u|^p - f_2$$

   on a compact oriented surface using your favourite method.