Inverse Hessian equations and positivity conditions on projective manifolds

Ved Datar

Indian Institute of Science

June 17, 2020

Outline

Outline

1. The Kähler cone
2. Demailly-Paun criteria
3. The inverse Hessian equations

Outline

Kähler manifolds

- Let M^n be a compact, complex manifold of dimension n.
- A $(1,1)$ form ω, locally given by

$$\omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta}$$

is called *Kähler* if it is

1. real $\iff \omega = \bar{\omega} \iff \{g_{\alpha\bar{\beta}}\}$ is a hermitian symmetric matrix.
2. positive $\iff \omega(\xi, \bar{\xi}) > 0$ for all $\xi = \sum_i \xi^i \frac{\partial}{\partial z^i} \in T^{1,0}(M) \iff \{g_{\alpha\bar{\beta}}\}$ is positive definite.
3. closed $\iff d\omega = 0 \iff$ at every point $p \in M$, there exists a system of holomorphic coordinates $(z^1, \cdots, z^n)$ centred at p such that

$$g_{\alpha\bar{\beta}}(z, \bar{z}) = \delta_{\alpha\bar{\beta}} + O(|z|^2).$$

Riemannian metric

- Recall that there is an endomorphism $J \in \mathrm{End}(TM)$, called the "complex structure" satisfying $J^2 = -id$. If $z^i = x^{2i-1} + \sqrt{-1}x^{2i}$, then

$$J\Big(\frac{\partial}{\partial x^{2i-1}}\Big) = \frac{\partial}{\partial x^{2i}}, \ J\Big(\frac{\partial}{\partial x^{2i}}\Big) = -\frac{\partial}{\partial x^{2i-1}}.$$

- A Kähler form ω then defines a natural Riemannian metric g on M:

$$g(X,Y) = \omega(X, JY).$$

- The volume form is then given by

$$dV = \frac{\omega^n}{n!}.$$

- In fact if $X \subset M$ is any complex sub-manifold of dimension p, $\omega\Big|_X$ is a Kähler metric on X, and

$$\mathrm{Vol}(X,\omega) = \int_X \frac{\omega^p}{p!}.$$

Examples of Kähler manifolds

- **Riemann surfaces:** Let $n = 1$. Then there always exists a Kähler form.
- **Complex flat space:** $\mathbb{C}^n$ with

$$\omega_{\mathbb{C}^n} := \frac{\sqrt{-1}}{2}\Big(dz^1 \wedge d\bar{z}^1 + \cdots + dz^n \wedge d\bar{z}^n\Big).$$

- **Complex projective space** $\mathbb{P}^n$**.** This is defined as $\mathbb{P}^n = \mathbb{C}^{n+1}/\mathbb{C}^*$, where $t \cdot (\xi^0, \cdots, \xi^n) = (t \cdot \xi^0, \cdots, t \cdot \xi^n)$. Then

$$\omega_{FS} := \sqrt{-1}\partial\bar{\partial} \log\Big(|\xi_0|^2 + \cdots + |\xi_n|^2\Big)$$

  is a Kähler metric on $\mathbb{P}^n$, called the *Fubini-Study* metric.
- **Projective varieties:** Let $M \subset \mathbb{P}^n$ be a complex sub-manifold. Then $\omega := \omega_{FS}\Big|_M$ defines a Kähler metric on M.
- **Tori:** Let $\Lambda \subset \mathbb{C}^n$ be a lattice. Then the Euclidean metric $\omega_{\mathbb{C}^n}$ (being translation invariant) induces a Kähler form ω_Λ on $M_\Lambda := \mathbb{C}^n/\Lambda$.
- (A non-example) **Hopf surface:** If $H = \mathbb{C}^2 \setminus \{(0,0)\}/(x,y) \sim (2x, 2y)$. This is not Kähler.

The Kähler cone

- ω a Kähler form $\implies$ (by being virtue of being closed and real) $[\omega]$ represents a cohomology class in $H^{1,1}_{\bar\partial}(M) \cap H^2(M,\mathbb{R})$.
- ($\sqrt{-1}\partial\bar\partial$-Lemma) $\omega' \in [\omega] \iff \omega' = \omega + \sqrt{-1}\partial\bar\partial\varphi$ for some $\varphi \in C^\infty(M,\mathbb{R})$.
- Conversely, we say that a class $\alpha \in H^{1,1}_{\bar\partial}(M) \cap H^2(M,\mathbb{R})$ is *positive* or *Kähler*, and write $\alpha > 0$, if it contains a Kähler metric.
- The Kähler cone $\mathcal{K}$ is defined to be

$$\mathcal{K} := \{\alpha \in H^{1,1}_{\bar\partial}(M) \cap H^2(M,\mathbb{R}) \mid \alpha > 0\}.$$

- **Fact:** $\mathcal{K}$ is an open, convex cone in the finite dimensional space $H^{1,1}_{\bar\partial}(M) \cap H^2(M,\mathbb{R})$. Convex cone simply means that $\alpha \in \mathcal{K} \implies t\alpha \in \mathcal{K}$ for all $t > 0$.

Question

Given a Kähler manifold M, how can one characterize it's Kähler cone $\mathcal{K}$?

Examples of the Kähler cone

- Let M be a Riemann surface. Then $H^{1,1}_{\bar\partial}(M) = H^2(M,\mathbb{R}) \cong \mathbb{R}$. Let ω be a Kähler form with $\int_M \omega = 1$. Then any other $\alpha \in H^{1,1}_{\bar\partial}(M)$ is given by $\alpha = t[\omega]$ for some $t \in \mathbb{R}$. If $\alpha > 0$, then there exists a $\omega' \in \alpha$ such that $\omega' > 0$. In particular, $\int_M \omega' = \int_M \alpha > 0$. So $t > 0$. Hence $\mathcal{K} = \mathbb{R}_+$.
- Now let $M_\Lambda = \mathbb{C}^n/\Lambda$. Hodge theory $\implies$

$$H^{1,1}_{\bar\partial}(M) \cap H^2(M,\mathbb{R}) = \Big\{ \Big[\sum_{i,j} a_{i\bar j} dz^i \wedge dz^{\bar j} \Big] \mid A := \{a_{i\bar j}\} \text{ is hermitian symmetric} \Big\}.$$

Hence

$$\mathcal{K} \cong \{A \in M_{n\times n}(\mathbb{C}) \mid A \text{ is a hermitian positive definite matrix}\}.$$

Ample line bundles

- Let $\pi : L \to M$ be a holomorphic line bundle i.e. $M = \cup_{\alpha=1}^{N} U_\alpha$, $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ holomorphic such that
$$L = \sqcup_\alpha U_\alpha \times \mathbb{C} \backslash \sim,$$
where $(x_\alpha, \lambda_\alpha) \sim (x_\beta, \lambda_\beta)$ if and only if $x_\alpha = x_\beta$ and $\lambda_\alpha = \tau_{\alpha\beta}(x_\alpha) \cdot \lambda_\beta$. So locally $L\Big|_{U_\alpha} = U_\alpha \times \mathbb{C}$, and $L_x := \pi^{-1}(x)$ is a one-dimensional $\mathbb{C}$ vector space.
- A *hermitian metric* h on L is a fiberwise hermitian metric $h(x)$ on each L_x which varies smoothly. Locally, it is specified by $h_\alpha \in C^\infty(U_\alpha, \mathbb{R}_+)$ such that for $x \in U_\alpha \cap U_\beta$, $h_\alpha(x) = |\tau_{\alpha\beta}(x)|^2 h_\beta(x)$.
- The *curvature* of h given by the form $\Theta_h := -\sqrt{-1}\partial\bar{\partial} \log h$ is a global closed $(1,1)$, purely imaginary form. In fact $\omega_h := \frac{\sqrt{-1}}{2\pi}\Theta_h$ is a global, closed, real $(1,1)$ form.
- The first Chern class of L is defined to be the cohomology class $c_1(L) := [\omega_h] \in H^{(1,1)}_{\bar{\partial}}(M) \cap H^2(M, \mathbb{R})$.
- **Fact:** $c_1(L) \in H^2(M, \mathbb{Z})$, and conversely any class $\alpha \in H^{1,1}_{\bar{\partial}}(M) \cap H^2(M, \mathbb{Z})$ is $c_1(L)$ for some line bundle L. The cohomology classes $H^{1,1}_{\bar{\partial}}(M) \cap H^2(M, \mathbb{Z})$ will henceforth be called integral classes.

The Seshadri-Nakai-Moishezon criteria

- We say that L is *ample* if $c_1(L) \in \mathcal{K} \iff$ there exists a hermitian metric h such that ω_h is a Kähler form.
- (Kodaira) If L is ample, then holomorphic sections of L^k given an embedding of M into $\mathbb{P}^{N_k}$ for $k >> 1$.

Theorem 1.1

Let M be projective.

1. *(Nakai-Moishezon) A line bundle $L \to M$ is ample if and only if for any sub-variety $V \subset M$,*
$$\int_V c_1(L)^{\dim V} > 0.$$
2. *(Seshadri) A line bundle $L \to M$ is ample if and only if for any $x \in M$, there exists a constant $\varepsilon(x) > 0$ such that for any curve $C \subset M$ passing through x,*
$$\int_C c_1(L) > \varepsilon(x)\mathrm{mult}_x C.$$

Some positivity classes

Let (M^n, χ) be a Kähler manifold.

- We have already seen the Kähler cone $\mathcal{K}$. We now introduce one more positivity classes.
- The class $\mathcal{P}$ is the class of classes numerically positive on analytic sets, ie.

$$\mathcal{P} := \{\alpha \in H^{1,1}_{\bar\partial}(M) \cap H^2(M, \mathbb{R}) \mid \int_Y \alpha^{\dim Y} > 0 \text{ for all subvarieties } Y \subset M\}.$$

Remark

It is tempting to imagine, especially given the Nakai criteria, that $\mathcal{K} = \mathcal{P}$. But this is wrong!

An example

Consider the torus $M_\Lambda = \mathbb{C}^n/\Lambda$. If $n \geq 2$, for a generic choice of Λ, M_Λ is not projective. Worse, it has no non-trivial sub-variety. Recall that

$$H^{1,1}_{\bar\partial}(M) \cap H^2(M,\mathbb{R}) = \Big\{ \Big[\sum_{i,j} a_{i\bar j} dz^i \wedge d\bar{z}^{j} \Big] \mid A := \{a_{i\bar j}\} \text{ is hermitian symmetric} \Big\}$$

$$\mathcal{K} = \{A \in M_{n\times n}(\mathbb{C}) \mid A \text{ is a hermitian positive definite matrix}\}.$$

On the other hand, since the only sub-variety is M itself,

$$\mathcal{P} = \{A \in M_{n\times n}(\mathbb{C}) \mid A \text{ is a hermitian matrix with } \det(A) > 0\}.$$

So $\mathcal{P} \subsetneq \mathcal{K}$.

Main theorems of Demailly-Paun

Theorem 2.1

The following are equivalent (TFAE).

1. α *is a Kähler class.*
2. $\alpha + t[\chi] \in \mathcal{P}$ *for all* $t > 0$.
3. *For every irreducible analytic set* $V \subset X$ *of pure dimension* p, *and for every* $k = 1, 2, \cdots, p$,
$$\int_V \alpha^k \wedge \chi^{p-k} > 0.$$

Remark

In the case of tori $M_\Lambda = \mathbb{C}^n/\Lambda$, *condition* (3) *is equivalent to the statement that*

$$\int_{M_\Lambda} \alpha^k \wedge \omega_{M_\Lambda}^{n-k} > 0,$$

which in turn is equivalent to saying that each minor of A, *the hermitian symmetric matrix representing* α, *is positive, that is* A *itself is positive definite. So the theorem is easily verified for tori.*

A generalization of the Nakai criteria for projective manifolds

Corollary 2.1

If M is projective, then $\mathcal{K} = \mathcal{P}$.

Proof.

It is enough to show that $\mathcal{P} \subset \mathcal{K}$. Suppose $M \subset \mathbb{P}^N$, and suppose $\alpha \in \mathcal{P}$, that is,

$$\int_V \alpha^k > 0$$

for all sub-varieties of dimension k. Let $\chi = c_1(H)$, where H is the restriction of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^N}(1)$. Then the corollary follows from the Theorem and the observation

$$\int_V \alpha^k \wedge \chi^{p-k} = \int_{V \cap H_1 \cap H_2 \cdots H_{p-k}} \alpha^k,$$

with hyperplanes $H_1, \cdots, H_{p-k}$ in general position. □

Outline of the proof of the theorem of Demailly-Paun in the projective case

The main aim is to show that (3) $\implies$ (1). Let $I = \{t \in [0,\infty) \mid \alpha + t[\chi] \in \mathcal{K}\}$. Then $t \in I$ if $t >> 1$. Let $t_0 = \inf I$. W. l. o. g. suppose $t_0 = 0$.
Goal: $0 \in I$.
The key new innovation of Demailly-Paun is the so-called mass concentration technique.

Proposition 2.1

Let (M,χ) be Kähler, and let α such that $\alpha + t[\chi] \in \mathcal{K}$ for all $t > 0$ and $\int_M \alpha^n > 0$. Then given any ample divisor Y, there exists a non-negative current $\Theta \in \alpha$ such that $\Theta \geq \beta_Y[Y]$ for some $\beta_Y > 0$ and $\Theta = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ for some $\omega_0 \in \alpha$ and $\varphi \in L^\infty(M)$.

- Here a $(1,1)$ current Θ is simply a linear functional on $\mathcal{A}^{(n-1,n-1)}(M)$.
- **Example:** If $Y = \sum a_i Y_i$ is a divisor, then that defines a $(1,1)$ current of integration by
$$\langle [Y], \eta \rangle = \sum_i a_i \int_{Y_i} \eta.$$
- We say that $\Theta \in \alpha$ if $\Theta = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ for some ω_0 a smooth form in α and $\varphi \in L^1_{loc}$.
- We say that $\Theta > 0$ if locally $\Theta = \sqrt{-1}\partial\bar{\partial}\theta$ such that θ is a plurisubharmonic function (ie. sub-harmonic when restricted to complex lines).

The main steps in the proof of Demailly-Paun

Assumption: $\alpha + t[\chi] \in \mathcal{K}$ for all $t > 0$ and for all $V \subset M$,

$$\int_V \alpha^k \wedge \chi^{\dim V - k} > 0.$$

Goal: $\alpha \in \mathcal{K}$.

- **Step-1:** (Mass concentration) There exists $\Theta \in \alpha$ such that $\Theta \geq \beta_Y[Y]$ for some $\beta_Y > 0$.
- **Step-2:** Then $\Xi = \Theta - \frac{\beta_Y}{2}[Y] + \frac{\beta_Y}{2}\chi_Y > 0$ on M. Moreover $\Xi \in \alpha$. Let $T = (1-\delta)\Xi \in (1-\delta)\alpha$. Then $T > 0$ on M.
- **Step-3:** (Gao Chen's new idea) For c, let $E_c(T) = \{x \in M \mid \nu(T, x) > c\}$ be the Lelong sub-level set, which is analytic by Siu. Suppose $Z = E_c(T)$ is smooth for some $c << 1$. Induction hypothesis, implies that Z has a Kähler metric in $\omega_Z \in \alpha\Big|_Z$. Extend it to a Kähler metric $\omega_U = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_U$ to some neighbourhood of U. Then glue this to a regularization of T on $M \setminus U'$ where $\overline{U'} \subset U$. This can be done if the Lelong number $c << 1$.
- **Step-4** In general Z is not smooth, and so take a resolution of singularities.

Outline of proof of mass concentration

- Y be cut-out by a section s of $[Y]$, and let h be a hermitian metric on Y. Let $\chi_t = \chi + A^{-1}\sqrt{-1}\partial\overline{\partial}\log(|s|_h^2 + t^2) > \chi/2$. Note that χ_t concentrates near Y as $t \to 0$.
- (Yau) $(1+t)\alpha$ Kähler $\implies$ there exists a unique $\omega_t \in (1+t)\alpha$ such that $\omega_t^n = c_t\chi_t^n$, where $c_t \geq c_0 > 0$.
- $\omega_{t_i} \rightharpoonup \Theta$ for some non-negative current Θ.
- Then one can show that for any neighbourhood U, there exists $\beta_U > 0$ such that

$$\int_{U\cap\{|s|_h^2<t^2\}} \omega_t \wedge \chi^{n-1} > \delta_U.$$

- Skoda extension + Support theorems from pluripotential theory $\implies \Theta \geq \beta_Y[Y]$ for some $\beta_Y > 0$.

Some remarks

- For non-projective case, Demailly Paun work on $\tilde{M} = M \times M$ and with Y as the diagonal $\Delta \subset \tilde{M}$ to obtain a (n, n) current $\tilde{\Theta} \in [\tilde{\alpha}^n]$, where $\tilde{\alpha} = \pi_1^*\alpha + \pi_2^*\alpha$, and then get the $(1, 1)$ K—'ahler current $\Theta = (\pi_1)_*(\tilde{\Theta} \wedge \pi_2^*\omega_0)$ on M. They then use an induction argument and a result from Paun's thesis - If T is a Kähler current in α and $\alpha\Big|_Z$ is Káhler on Z for every irreducible analytic set $Z \subset M$, then α is Kähler.
- By a result of Boucksom, one can show that if $\alpha + t\chi \in \mathcal{K}$ for all $t > 0$ and $\int_M \alpha > 0$, then there exists a Kähler current in α with analytic singularities.
- (Collins-Tosatti)

$$\Big\{\text{Non Kähler locus of } \alpha\Big\} = \Big\{\text{Null locus of } \alpha\Big\}.$$

1 The Kähler cone

2 Demailly-Paun criteria

3 The inverse Hessian equations

Mabuchi functional and the cscK problem

A question of central interest in Kähler geometry is to contruct constant scalar curvature Kähler (cscK) metrics.

Conjecture

(Yau-Tian-Donaldson) Let $\alpha \in \mathcal{K}$. There exists a Kähler form $\omega \in \alpha$ whose scalar curvature $\mathbf{s}_\omega$ is constant if (and only if) the pair (M, α) is "stable".

- Let ω_0 be a reference metric in α, and let

$$\mathcal{H}_\alpha := \{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

- (Mabuchi energy, Chen) A metric ω_φ is cscK if and only if it is a smooth critical point of the functional

$$K(\varphi) = \int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\frac{\omega_\varphi^n}{n!} + J_{-\mathrm{Ric}(\omega_0)}(\varphi),$$

where for any closed, real $(1,1)$ form χ, J_χ is defined by the variational formula

$$\delta J_\chi(\varphi) := \int_M \delta\varphi\Big(c_{n-1}\chi \wedge \frac{\omega_{\varphi^{n-1}}}{(n-1)!} - \frac{\omega_\varphi^n}{(n-1)!}\Big).$$

The J-equation

- From the previous slide:

$$\delta J_\chi(\varphi) := \int_M \delta\varphi\Big(c_{n-1}\chi \wedge \frac{\omega_{\varphi^{n-1}}}{(n-1)!} - \frac{\omega_\varphi^n}{(n-1)!}\Big).$$

- Clearly a metric ω_φ is a critical point if and only if it satisfies the so-called J-equation:

$$\omega_\varphi^n = c_{n-1}\chi \wedge \omega_\varphi^{n-1}.$$

- From now on we assume that $\chi > 0$. Then the above functional is "convex" on $\mathcal{H}_\alpha$. Moreover, if there is a solution to the J-equation, then J_χ is "proper".
- In particular, studying the J-equation on manifolds of general type (so that one can choose ω_0 with $Ric(\omega_0) < 0$) helps in constructing cscK metrics.

Some necessary conditions

- Recall the J-equation:

$$\omega_\varphi^n = c_{n-1}\chi \wedge \omega_\varphi^{n-1}.$$

- A trivial necessary condition is obtained by integrating both sides, ie.

$$c_{n-1} = \frac{\int_M \omega_0^n}{\int_M \chi \wedge \omega_0^{n-1}}.$$

- If $n = 2$, then the completing squares, the equation is equivalent to

$$\Big(\omega_0 - \frac{c_1}{2}\chi + \sqrt{-1}\partial\bar{\partial}\varphi\Big)^2 = \chi^2.$$

- By Yau's solution to the Calabi conjecture a necessary condition is that $[\omega_0] - \frac{c_1}{2}[\chi] > 0$, that is if we can choose a metric $\omega_0 \in [\omega_0]$ such that $\omega_0 - \frac{c_1}{2}\chi > 0$.

A necessary and sufficient condition

- More generally, if $0 < \lambda_1 \cdots < \lambda_n$ are the eigenvalues of $\chi^{-1}\omega_\varphi$, the equation is
$$\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} = \frac{n}{c_{n-1}}.$$
- So a necessary condition is that for any j,
$$\sum_{i \neq j} \frac{1}{\lambda_i} < \frac{n}{c_{n-1}} \iff n\omega_\varphi^{n-1} - c_{n-1}(n-1)\chi \wedge \omega_\varphi^{n-2} > 0.$$

Theorem 3.1 (Song-Weinkove [5], 2009)

Let (M, χ) be a Kähler manifold and ω_0 another Kähler metric and $c_{n-1} = \frac{\int_M \omega_0^n}{\int_M \chi \wedge \omega_0^{n-1}}$. TFAE.

1. *There exists a Kähler metric $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ such that*
$$\begin{cases} \omega_\varphi^n = c_{n-1}\chi \wedge \omega_\varphi^{n-1} \\ n\omega_\varphi^{n-1} - c_{n-1}(n-1)\chi \wedge \omega_\varphi^{n-2} > 0. \end{cases} \quad (3.1)$$
2. *There exists a Kähler metric $\hat{\omega}_0 \in [\omega_0]$ satisfying the cone condition*
$$n\hat{\omega}_0^{n-1} - c_{n-1}(n-1)\chi \wedge \hat{\omega}_0^{n-2} > 0.$$

A numerical criteria, a la Demailly-Paun

Theorem 3.2 (Gao Chen [1])

Let (M^n, χ) be a Kähler manifold and ω_0 another Kähler metric. Let $c_{n-1} = \frac{\int_M \omega_0^n}{\int_M \chi \wedge \omega_0^{n-1}}$. TFAE.

1. *There exists a Kähler metric $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ satisfying*

$$\begin{cases} \omega_\varphi^n = c_{n-1}\chi \wedge \omega_\varphi^{n-1} \\ n\omega^{n-1} - c_{n-1}(n-1)\chi \wedge \omega_\varphi^{n-2} > 0. \end{cases} \tag{3.2}$$

2. *There exists a Kähler metric $\hat{\Omega} \in [\Omega_0]$ satisfying the cone condition*

$$n\hat{\omega}_0^{n-1} - c_{n-1}(n-1)\chi \wedge \hat{\omega}_0^{n-2} > 0.$$

3. *There exists an $\varepsilon > 0$ such that for any p-dimensional sub-variety $V \subset X$,*

$$\int_V \left(n\omega_0^p - c_{n-1}p\chi \wedge \omega_0^{p-1}\right) > \varepsilon \int_V n\omega_0^p.$$

Note that the final condition only depends on the cohomology classes $[\omega_0]$ and $[\chi]$.

More general inverse Hessian equations

More generally, Szekelyhidi, and others before him, considered the equation

$$\omega_\varphi^n = c_k \chi^{n-k} \wedge \omega_\varphi^k,$$

and proved that a solution exists if and only if the cone condition is met for some $\hat{\omega} \in [\omega_0]$,

$$n\hat{\omega}^n - c_k k \chi^{n-k} \wedge \hat{\omega}^{k-1} > 0$$

is satisfied. Szekelyhidi then made the following conjecture:

Conjecture (Szekelyhidi [6])

The above equation has a solution if and only if for every subvariety V of codimension $n-k \leq p \leq n-1$, the following inequality holds.

$$\int_V \binom{n}{p} [\omega_0]^{n-p} - \int_V c \binom{k}{p} \chi^k [\omega_0]^{n-p-k} > 0.$$

Generalized inverse Hessian equations

Let $c_1, \cdots, c_{n-1}$ be non-negative real numbers, such that at least one is positive. Suppose

$$\int_M \omega_\varphi^n = \int_M \sum_{k=1}^{n-1} c_k \chi^{n-k} \omega_\varphi^k.$$

s

Theorem (D.–Pingali [4], 2020)

Let M be a projective manifold, and χ, ω_0 be Kähler forms. TFAE:

1. *The generalised Monge-Ampère equation has a solution $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ satisfying*

$$\begin{cases} \omega_\varphi^n = \sum_{k=1}^{n-1} c_k \chi^{n-k} \omega_\varphi^k, \\ n\omega_\varphi^{n-1} - \sum_{k=1}^{n-1} c_k k \chi^{n-k} \omega_\varphi^{k-1} > 0. \end{cases} \tag{3.3}$$

2. *(Cone condition) There exists a Kähler metric $\hat{\omega}_0 \in [\omega_0]$ satisfying the cone condition, i.e.,*

$$n\hat{\omega}_0^{n-1} - \sum_{k=1}^{n-1} c_k k \chi^{n-k} \hat{\omega}_0^{k-1} > 0.$$

3. *(Uniform stability condition) There exists a constant $\varepsilon > 0$ such that for all G-invariant subvarieties $V \subset M$ of co-dimension p, we have*

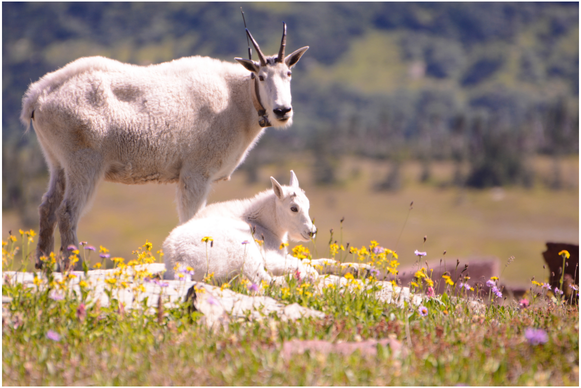
$$\int_V \left(\binom{n}{p} \omega_0^{n-p} - \sum_{k=p}^{n-1} c_k \binom{k}{p} \chi^{n-k} \wedge \omega_0^{k-p} \right) > \varepsilon \binom{n}{p} \int_V \omega_0^{n-p}.$$

Some remarks

- We obtain an equivariant version. In particular, for toric manifolds, one needs to only check the numerical criteria on torus invariant sub-varieties. For J-equation, this recovers results of Collins-Szekelyhidi.
- We believe that the uniformity can be relaxed (ie. we can prove the same theorem with $\varepsilon = 0$), and this is a work in progress.

Future directions

1. One can try to prove this theorem for non-projective Kähler manifolds. The problem is that one needs to solve an appropriate PDE on $M \times M$. In our case, it is not clear what this PDE should be.
2. One can instead also look at Hessian equations (ie. $\omega_\varphi^k \wedge \chi^{n-k} = a_k \chi^n$).
   - Here Kähler condition may have to be relaxed (think of the simplest case where $\mathrm{tr}_\chi \omega_\varphi = a_1$, this is equivalent to solving $\Delta_\chi \varphi = a_1 - \mathrm{tr}_\chi \omega_0$, and then even if ω_0 is Kähler, ω_φ need not be Kähler).
   - But one can come up with some obstructions in terms of cone conditions and numerical criteria, and there are corresponding conjectures (for Laplace equation this is simply $n \int \omega_0 \wedge \chi^{n-1} = a_1$).
3. One can attempt to obtain a Collins-Tosatti type theorem.
   - For instance, the null locus can be defined to be the "smallest" set where the cone condition fails. Then a natural question is whether this is an analytic set, and if so, is it the "largest" set on which the numerical criteria fails.
   - For this, maybe one has to obtain an analog of the Boucksom result, which itself might be hard at this point. That is, if α is Kähler and $\alpha + t\chi$ has a metric satisfying the cone condition, then is there a current $T \in \alpha$ satisfying the cone condition, but having "mild singularities".



Thank You for your attention!

Select references

- G. Chen. On J-equation. arXiv : 1905.10222.
- Collins, T. and Tosatti, V. "Kähler currents and null loci", *Invent. Math.*, n.0 3. (2015): 1167–1198
- Datar, V. and Pingali, V *A numerical criterion for generalised Monge-Ampere equations on projective manifolds*, arXiv:2006.01530.
- Demailly, J-P and Paun, M. "Numerical characterization of the Kähler cone of a compact Kähler manifold". *Ann. Math.* (2004) : 1247-1274.
- J. Song, and B. Weinkove. "The degenerate J-flow and the Mabuchi energy on minimal surfaces of general type." *Univ. Iagellonicae Acta Math.* 50 : 89-106.
- G. Székelyhidi. "Fully non-linear elliptic equations on compact Hermitian manifolds." *J. Differential Geom.* 109(2)(2018) : 337-378.