GEOMETRIC INVARIANTS FOR A CLASS OF SUBMODULES OF ANALYTIC HILBERT MODULES

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Abstract. Let $\Omega \subseteq \mathbb{C}^m$ be a bounded connected open set and $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ be an analytic Hilbert module, i.e., the Hilbert space $\mathcal{H}$ possesses a reproducing kernel K, the polynomial ring $\mathbb{C}[\underline{z}] \subseteq \mathcal{H}$ is dense and the point-wise multiplication induced by $p \in \mathbb{C}[\underline{z}]$ is bounded on $\mathcal{H}$. We fix an ideal $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$ and let $[\mathcal{I}]$ denote the completion of $\mathcal{I}$ in $\mathcal{H}$. Let $X : [\mathcal{I}] \to \mathcal{H}$ be the inclusion map. Thus we have a short exact sequence of Hilbert modules $0 \longrightarrow [\mathcal{I}] \xrightarrow{X} \mathcal{H} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0$, where the module multiplication in the quotient $\mathcal{Q} := [\mathcal{I}]^\perp$ is given by the formula $m_p f = P_{[\mathcal{I}]^\perp}(pf)$, $p \in \mathbb{C}[\underline{z}]$, $f \in \mathcal{Q}$. The analytic Hilbert module $\mathcal{H}$ defines a subsheaf $\mathcal{S}^{\mathcal{H}}$ of the sheaf $\mathcal{O}(\Omega)$ of holomorphic functions defined on Ω. For any open $U \subset \Omega$, it is obtained by setting

$$\mathcal{S}^{\mathcal{H}}(U) := \Big\{ \sum_{i=1}^{n} (f_i|_U) h_i : f_i \in \mathcal{H}, h_i \in \mathcal{O}(U), n \in \mathbb{N} \Big\}.$$

This is locally free and naturally gives rise to a holomorphic line bundle on Ω. However, in general, the sheaf corresponding to the sub-module $[\mathcal{I}]$ is not locally free but only coherent.

Building on the earlier work of S. Biswas, a decomposition theorem is obtained for the kernel $K_{[\mathcal{I}]}$ along the zero set $V_{[\mathcal{I}]} := \{z \in \mathbb{C}^m : p(z) = 0, p \in [\mathcal{I}]\}$: There exists anti-holomorphic functions $F^1, \ldots, F^t : V_{[\mathcal{I}]} \to [\mathcal{I}]$ such that

$$K_{[\mathcal{I}]}(\cdot, w) = \overline{p_1(u)} F_0^1 + \cdots \overline{p_t(u)} F_0^t,\ u \in \Omega_w,$$

in some neighbourhood Ω_w of each fixed but arbitrary $w \in V_{[\mathcal{I}]}$ for some anti-holomorphic functions $F_0^1, \ldots, F_0^t : \Omega_w \to [\mathcal{I}]$ extending the functions $F^1, \ldots, F^t$. The anti-holomorphic functions $F^1, \ldots, F^t$ are linearly independent on $V_{[\mathcal{I}]}$, therefore define a rank t holomorphic Hermitian vector bundle on it. This gives rise to complex geometric invariants for the pair $([\mathcal{I}], \mathcal{H})$.

Next, the functions $F^1, \ldots, F^t : V_{[\mathcal{I}]} \to [\mathcal{I}]$ are explicitly determined for pairs $([\mathcal{I}], \mathcal{H})$ after making certain assumptions. These cover a large class of examples.

Localising the modules $[\mathcal{I}]$ and $\mathcal{H}$ at $w \in \Omega$, we obtain the localization $X(w)$ of the module map X. The localizations are nothing but the quotient modules $[\mathcal{I}]/[\mathcal{I}]_w$ and $\mathcal{H}/\mathcal{H}_w$, where $[\mathcal{I}]_w$ and $\mathcal{H}_w$ are the maximal sub-modules of functions vanishing at w. These clearly define anti-holomorphic line bundles $E_{[\mathcal{I}]}$ and $E_{\mathcal{H}}$, respectively, on $\Omega \setminus V_{[\mathcal{I}]}$. However, there is a third line bundle, namely, $\mathrm{Hom}(E_{\mathcal{H}}, E_{[\mathcal{I}]})$ defined by the anti-holomorphic map $X(w)^*$. The curvature of a holomorphic line bundle $\mathcal{L}$ on Ω, computed with relative to a holomorphic frame γ is given by the formula

$$\mathcal{K}_{\mathcal{L}}(z) = \sum_{i,j=1}^{m} \tfrac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \|\gamma(z)\|^2 dz_i \wedge d\bar{z}_j.$$

It is a complete invariant for the line bundle $\mathcal{L}$. Now, consider the alternating sum

$$\mathcal{A}_{[\mathcal{I}],\mathcal{H}}(w) := \mathcal{K}_X(w) - \mathcal{K}_{[\mathcal{I}]}(w) + \mathcal{K}_{\mathcal{H}}(w) = 0,\ w \in \Omega \setminus V_{[\mathcal{I}]},$$

where $\mathcal{K}_X$, $\mathcal{K}_{[\mathcal{I}]}$ and $\mathcal{K}_{\mathcal{H}}$ denote the curvature $(1,1)$ form of the line bundles E_X, $E_{[\mathcal{I}]}$ and $E_{\mathcal{H}}$, respectively. Thus it is an invariant for the pair $([\mathcal{I}], \mathcal{H})$. However, when $\mathcal{I}$ is principal, by taking distributional derivatives $\mathcal{A}_{[\mathcal{I}],\mathcal{H}}(w)$, extends to all of Ω as a $(1,1)$ current. Consider the following diagram of short exact sequence of Hilbert modules:

$$\text{(1)}\quad \begin{array}{ccccccccc} 0 & \longrightarrow & [\mathcal{I}] & \xrightarrow{X} & \mathcal{H} & \xrightarrow{\pi} & \mathcal{Q} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle L} & & \big\downarrow & & \\ 0 & \longrightarrow & [\widetilde{\mathcal{I}}] & \xrightarrow{\widetilde{X}} & \widetilde{\mathcal{H}} & \xrightarrow{\tilde{\pi}} & \widetilde{\mathcal{Q}} & \longrightarrow & 0, \end{array} \qquad \text{(2)}\quad \begin{array}{ccc} [\mathcal{I}] & \xrightarrow{X} & \mathcal{H} \\ \big\downarrow & & \big\downarrow{\scriptstyle L} \\ [\widetilde{\mathcal{I}}] & \xrightarrow{\widetilde{X}} & \widetilde{\mathcal{H}} \end{array}$$

It is shown that if $\mathcal{A}_{[\mathcal{I}],\mathcal{H}}(w) = \mathcal{A}_{[\widetilde{\mathcal{I}}],\widetilde{\mathcal{H}}}(w)$, then $L|_{[\mathcal{I}]}$ makes the second diagram commute. Hence if L is bijective, then $[\mathcal{I}]$ and $[\widetilde{\mathcal{I}}]$ are equivalent as Hilbert modules. It follows that the alternating sum is an invariant for the "rigidity" phenomenon.