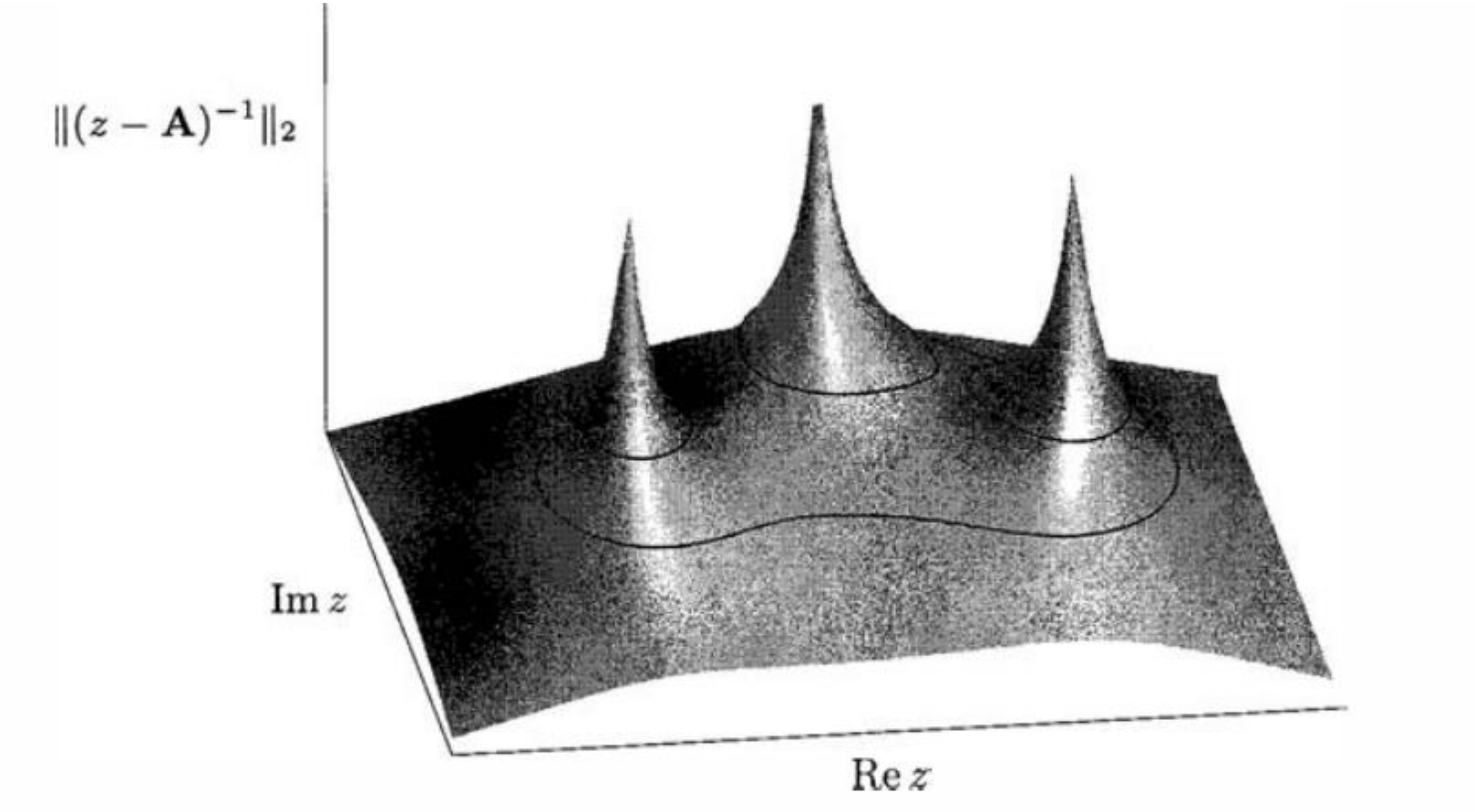
Quantitative Diagonalizability

Part I: Three Measures of Nonnormality

ϵ −pseudospectrum

$$\Lambda_\epsilon(M) \coloneqq \{z \in \mathbb{C} : ||(z - M)^{-1}|| \geq \epsilon^{-1}\}$$



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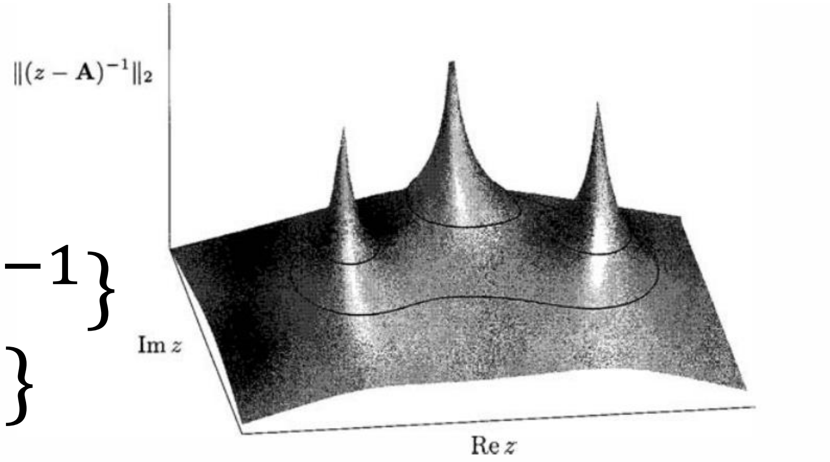

$$\Lambda_\epsilon(M) \coloneqq \{z \in \mathbb{C} : \ ||(z-M)^{-1}|| \geq \epsilon^{-1}\}$$
$$= \{z \in \mathbb{C} : \sigma_n(z-M) \leq \epsilon \}$$
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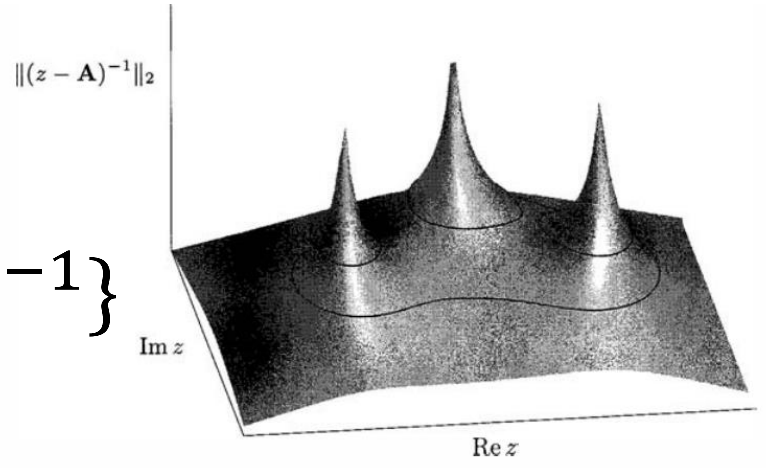
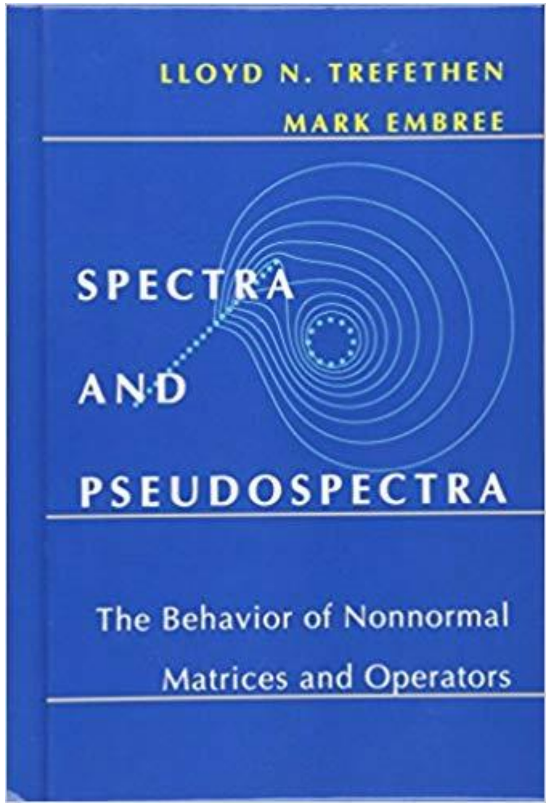
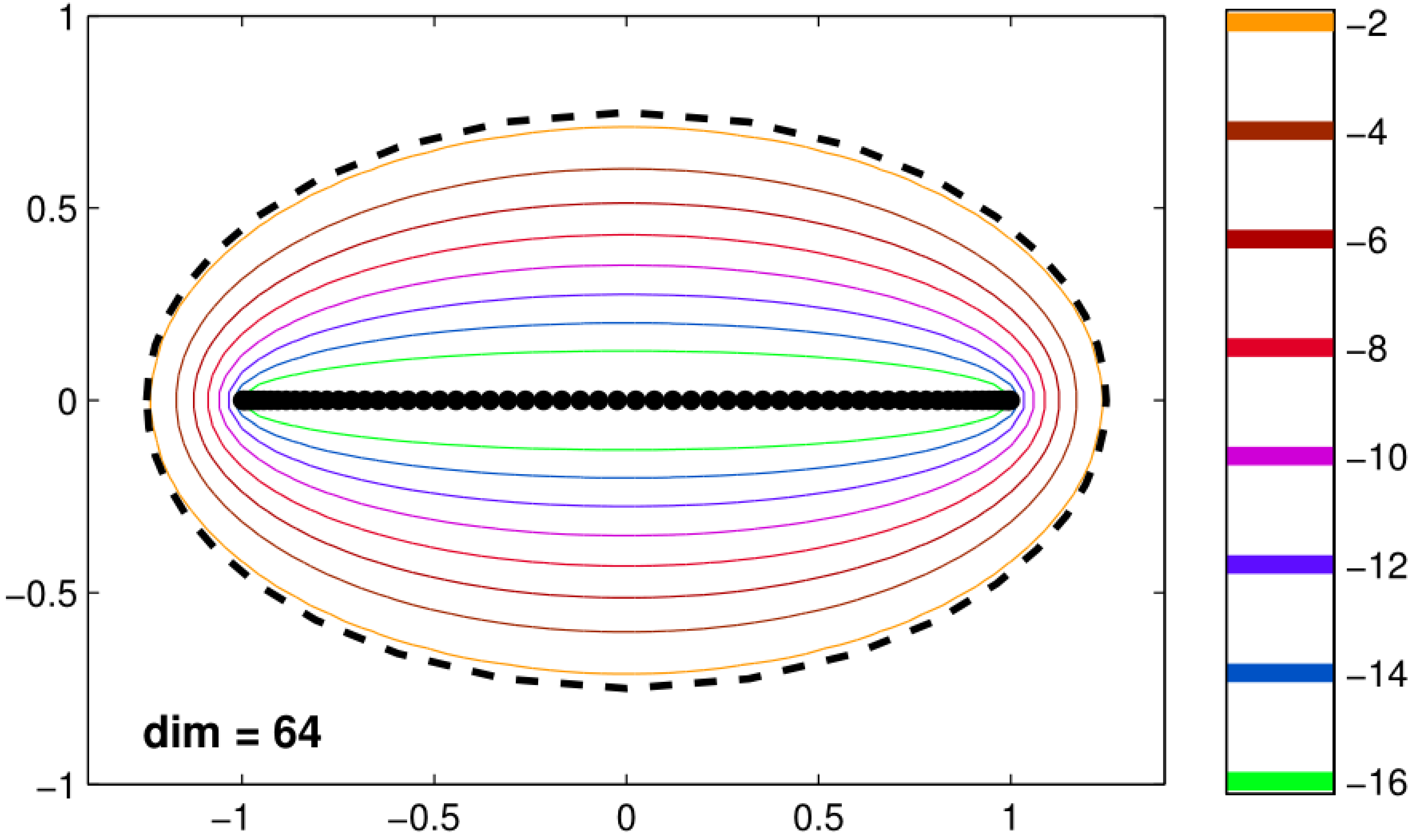


For normal matrices, $\Lambda_\epsilon(M) = \Lambda_0(M) + D(0, \epsilon)$

(a) normal (b) nonnormal



Pseudospectrum of Toeplitz Example



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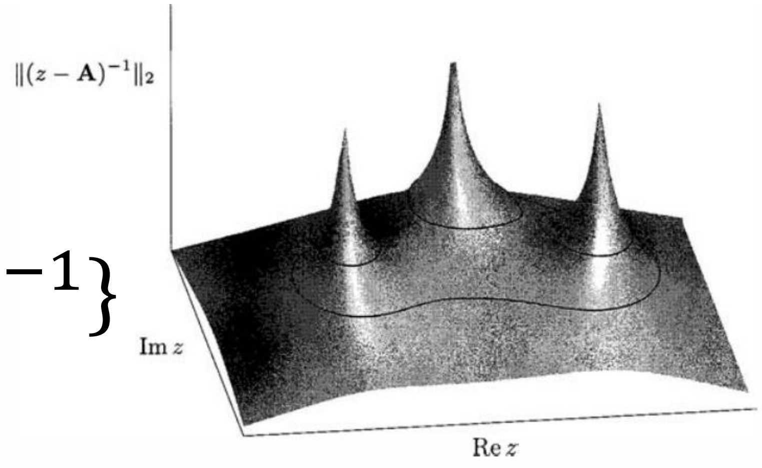

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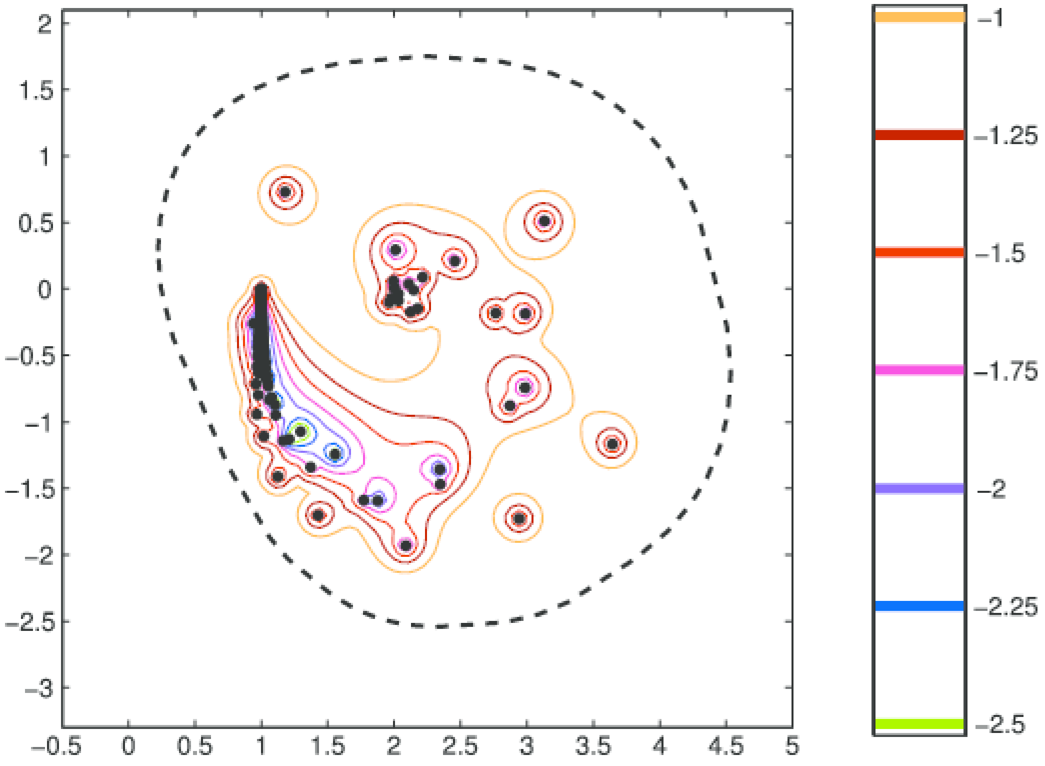
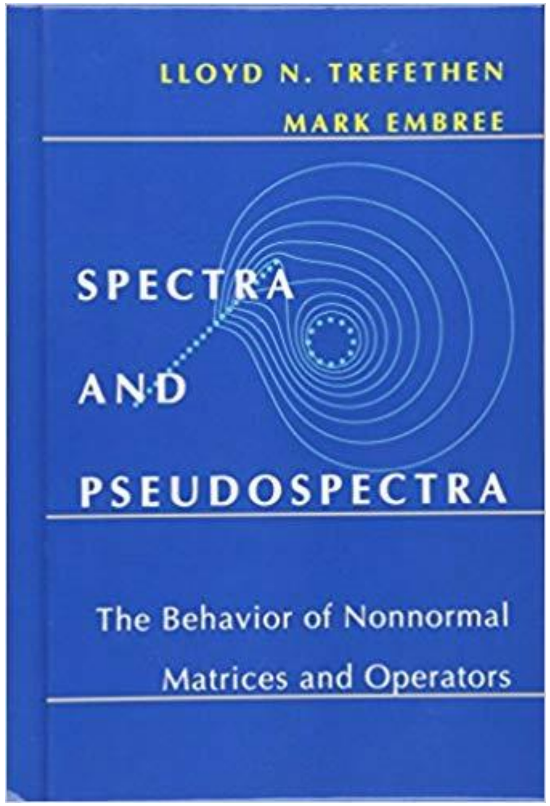
e.g. discretization of pde from acoustics:





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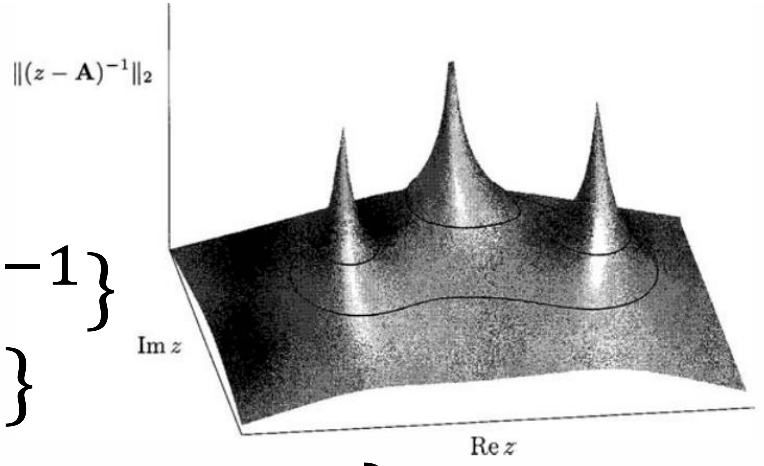

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[Bauer-Fike]: $\Lambda_\epsilon(M) \subset \Lambda_0(M) + \kappa_e(M) D(0,\epsilon)$

For distinct eigs $\Lambda_\epsilon(M) = \Lambda_0(M) + \cup_i D(\lambda_i, \kappa(\lambda_i)\epsilon) + o(\epsilon)$

Part II: Davies’ Conjecture

(with Jess Banks, Archit Kulkarni, Satyaki Mukherjee)

Diagonalization

$A \in \mathbb{C}^{n\times n}$ is **diagonalizable** if $A = VDV^{-1}$ for invertible V, diagonal D.

Every matrix is a limit of diagonalizable matrices.

Let $\kappa_e(A) \coloneqq ||V|| \cdot ||V^{-1}||$ be the **eigenvector condition number** of A**.**

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$\kappa_e(A) = 1$ for normal,
∞ for nondiagonalizable

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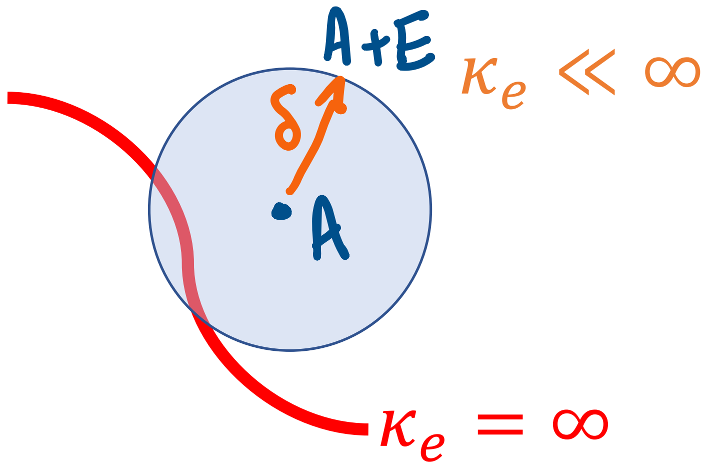

Question: Given a matrix A and $\delta > 0$, what is $\min\{\kappa_e(A+E): ||E|| \leq \delta\}$?

Motivation: Computing Matrix Functions

Problem. Compute $f(A)$ for analytic function f, e.g. $f(z) = e^z, z^p$.

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e.g. $n \times n$ Toeplitz, n=100:

$$\mathbf{A} = \begin{bmatrix} 0 & 1/2 & & \\ -2 & 0 & \ddots & \\ & \ddots & \ddots & 1/2 \\ & & -2 & 0 \end{bmatrix}$$

$$\kappa_e(A) = 2^{n-1} \approx 10^{30}$$

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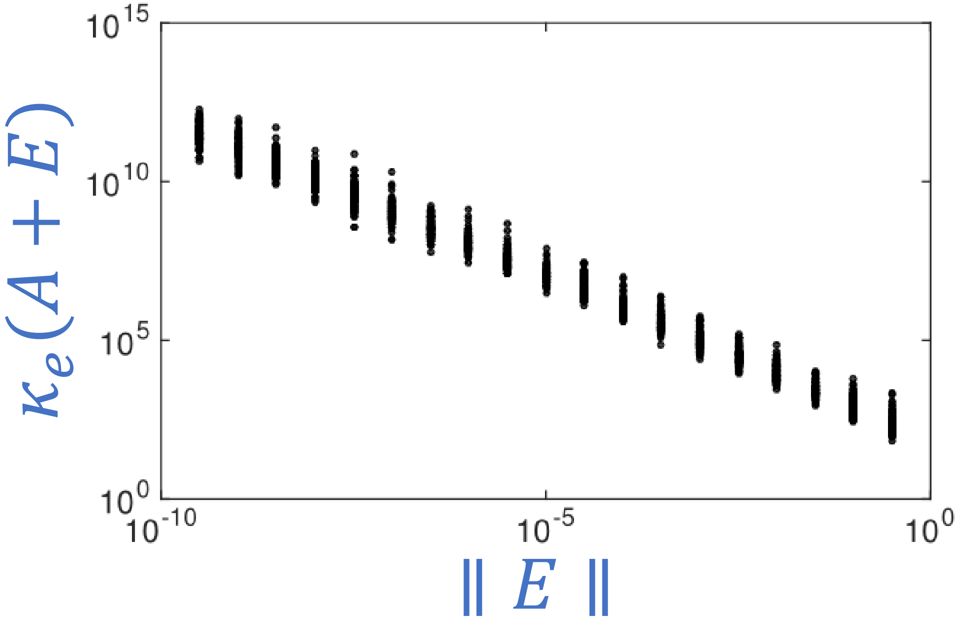

$E \sim$Gaussian

experiment by M. Embree

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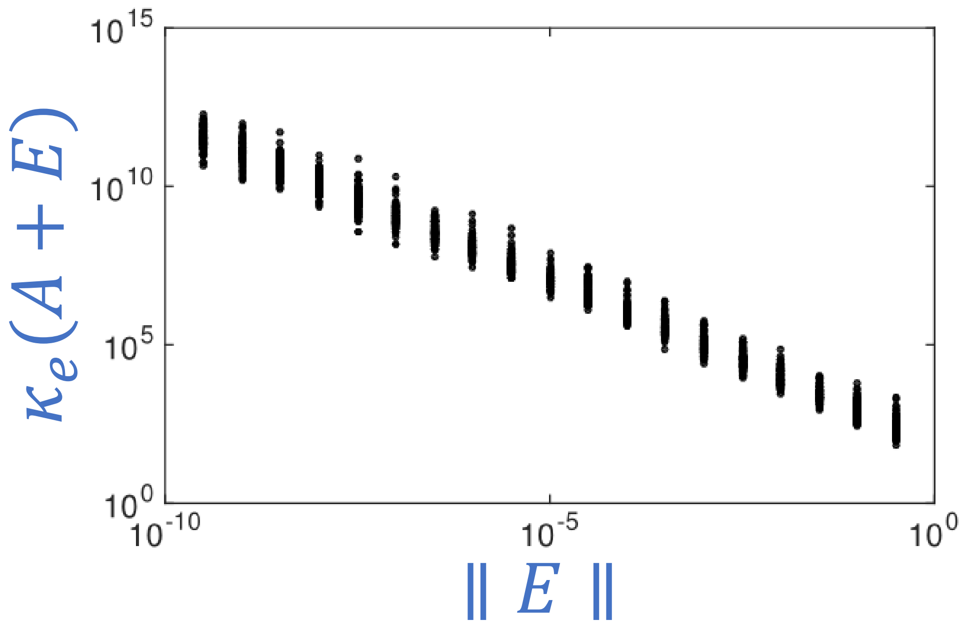

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Empirically: A is **close to** a matrix with much better κ_e.

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Vol. 29, No. 4, pp. 1051–1064

APPROXIMATE DIAGONALIZATION*

E. B. DAVIES†

Idea. Approximate $f(A)$ by $f(A+E)$ for some small E.

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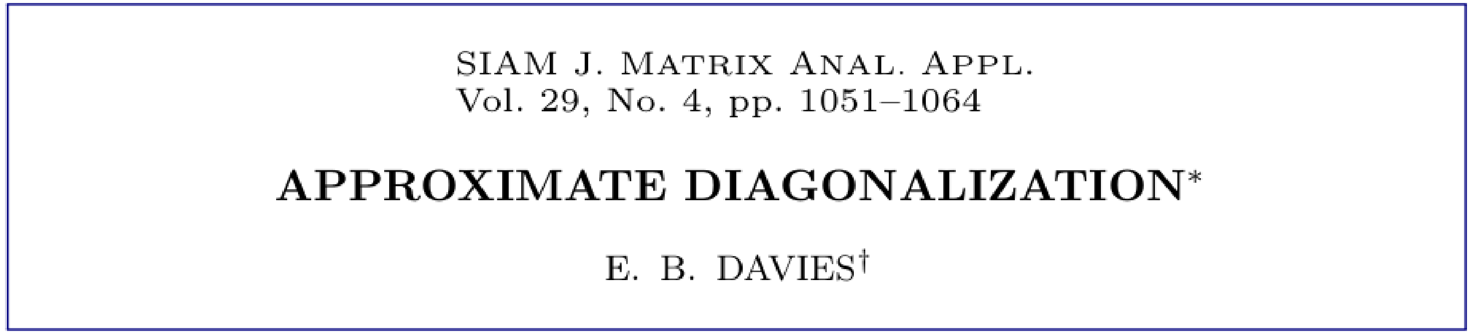

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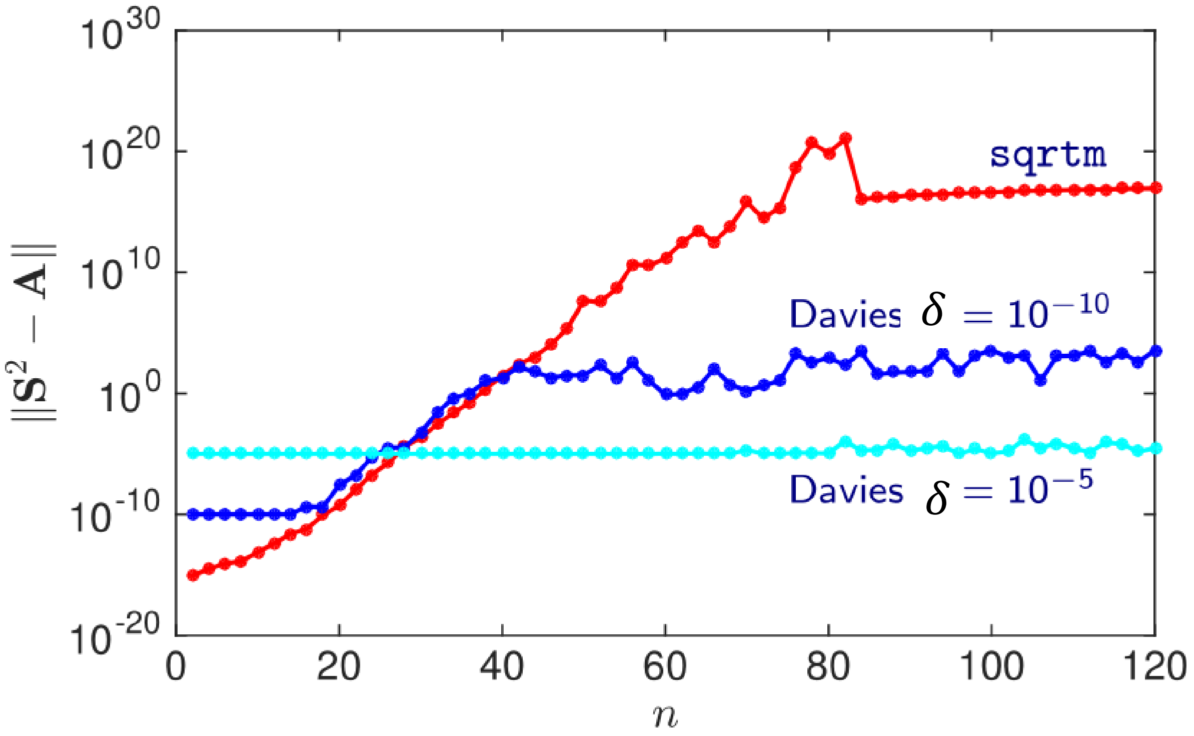

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Approximate Diagonalization

Theorem. [Davies'06] For every $A \in \mathbb{C}^{n\times n}$ with $||A|| \leq 1$ and $\delta \in (0,1)$ there is a perturbation E such that

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[Davies'06]: true for $n=3$ and for special case $A = J_n$, with $C_n = 2$.

Main Result

Theorem A. For every $A \in \mathbb{C}^{n \times n}$ with $||A|| \leq 1$ and $\delta \in (0,1)$ there is a perturbation E such that

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Implied by a stronger probabilistic result on **eigenvalue condition numbers.**

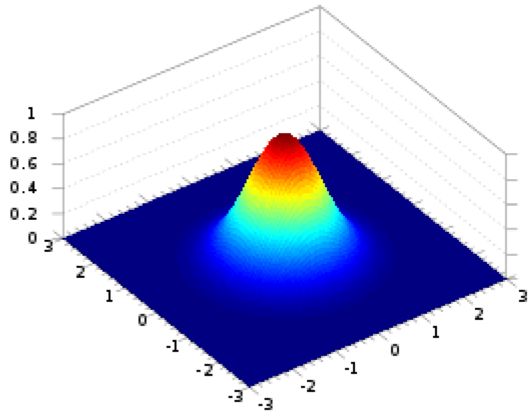
Probabilistic Analysis of κ_i

Theorem B. Assume $||A|| \leq 1$ and let G have i.i.d. *complex* standard Gaussian entries. Let $\lambda_1, \dots \lambda_n$ be the eigenvalues of $A + \gamma G$.

Probabilistic Analysis of κ_i

Theorem B. Assume $||A|| \leq 1$ and let G have i.i.d. *complex* standard Gaussian entries. Let $\lambda_1, \ldots \lambda_n$ be the eigenvalues of $A + \gamma G$

$z = x + iy$ where $x, y \sim N\left(0, \frac{1}{2}\right)$



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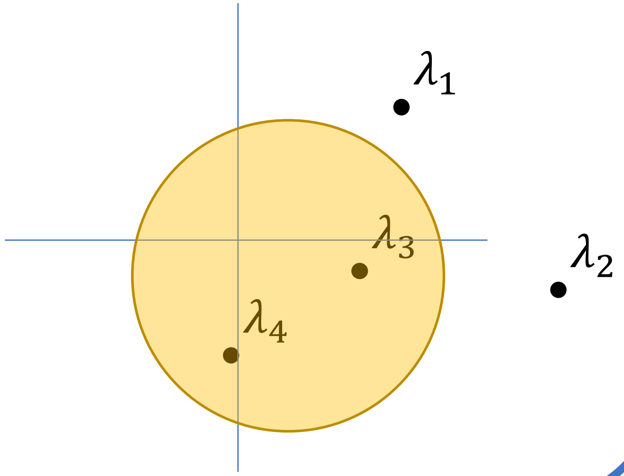
$$\mathbb{E} \sum_{\lambda_i \in B} \kappa^2(\lambda_i) \leq \frac{n}{\pi \gamma^2} \cdot vol(B)$$



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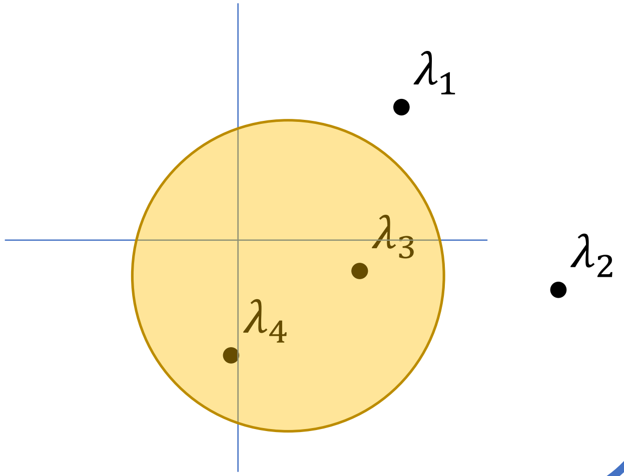

cf. Precise asymptotic results for $A = 0$ [Chalker-Mehlig'98,…Bourgade-Dubach'18,Fyodorov'18]

and A =Toeplitz [Davies-Hager'08,…Basak-Paquette-Zeitouni'14-18, Sjostrand-Vogel'18]

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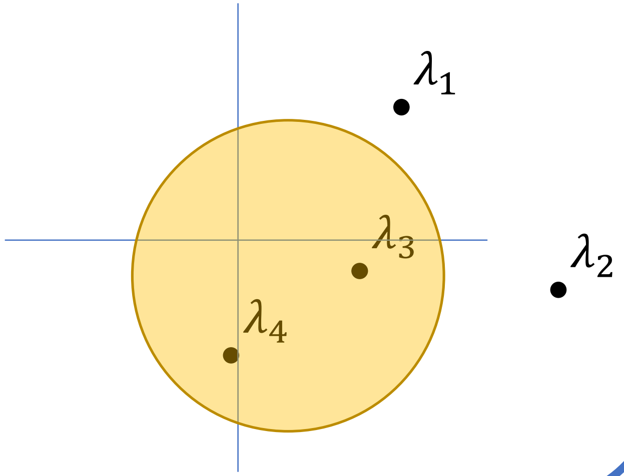

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Remark: Bourgade-Dubach implies that Theorem B is sharp for $A = 0$

Implication B->A

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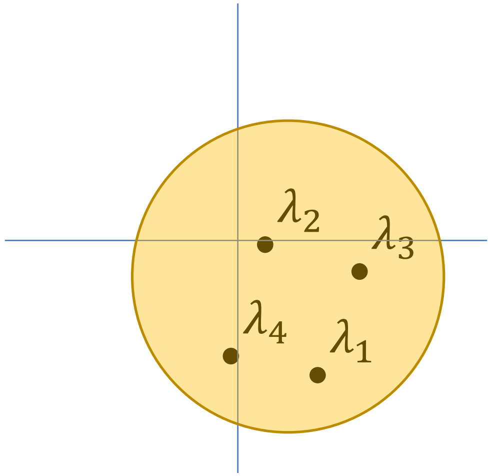

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$\delta \asymp \gamma\sqrt{n}$

□

Proof of Theorem B

1. Area of the pseudospectrum

Lemma 1: If M has distinct eigenvalues then for every open B:

$$\pi \sum_{\lambda_i \in B} \kappa(\lambda_i)^2 = \lim_{\epsilon \to 0} \frac{vol(\Lambda_\epsilon(M) \cap B)}{\epsilon^2}$$

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Theorem[Sankar-Spielman-Teng'06]: For any real $n \times n$ matrix M, and G with i.i.d. real $N(0,1)$ entries:

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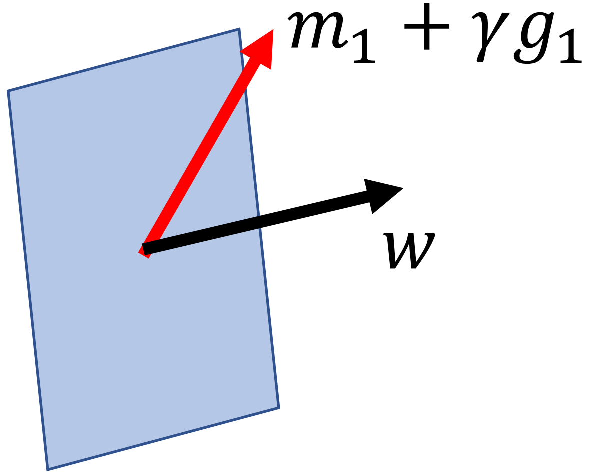

Proof Idea:

Let $M + \gamma G$ have columns $m_i + \gamma g_i$,

Let $S = span\{m_i + \gamma g_i\}_{i>2}$

$$\mathbb{P}[dist(m_1 + \gamma g_1, S) \le \epsilon] = \mathbb{P}[|\langle m_1 + \gamma g_1, w\rangle| \le \epsilon]$$
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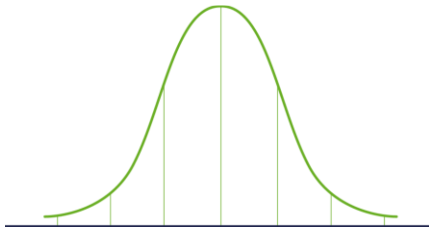
Orthogonal invariance

anticoncentration



2’. Complex Anticoncentration

Lemma 2. For any complex $n \times n$ matrix M, and G with i.i.d. complex $N(0,1_{\mathbb{C}})$ entries:

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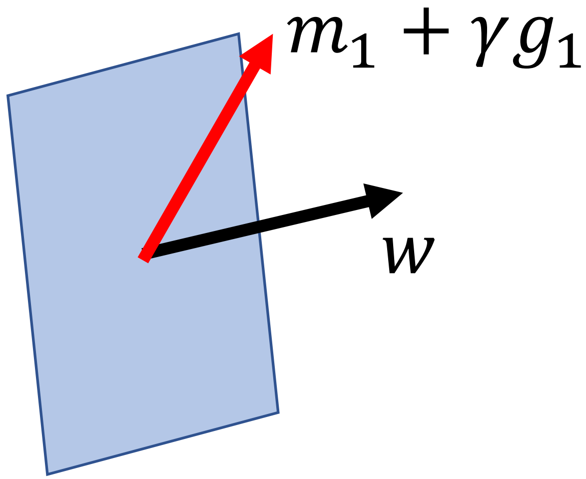

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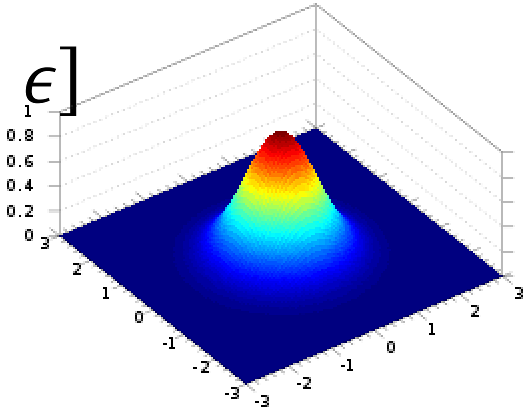

Unitary invariance

anticoncentration

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Cf. [Edelman'88] M=0

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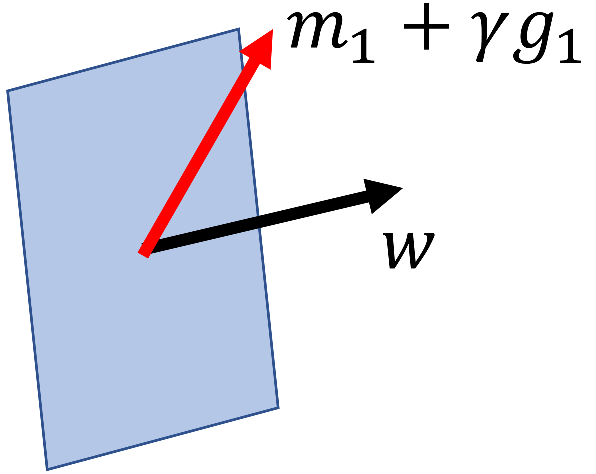

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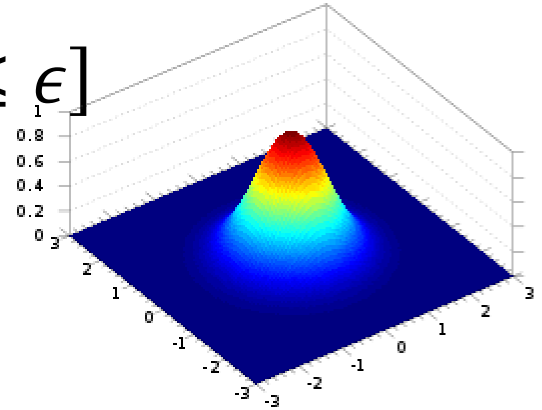

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Lemma 3. For every fixed ball B, for every $\epsilon > 0$:

$$\mathbb{E} vol(\Lambda_\epsilon(A + \gamma G) \cap B) \leq \frac{n\epsilon^2}{\gamma^2} \cdot vol(B)$$

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4. Expected Limiting Area of the Pseudospectrum

Define the function

$$f_\epsilon(G) := vol(\Lambda_\epsilon(A + \gamma G) \cap B)/\epsilon^2$$

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Lemma 2 shows that

$$\liminf_{\epsilon \to 0} \mathbb{E} f_\epsilon(G) \leq n/\gamma^2$$

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Define the function

$$f_\epsilon(G) := vol(\Lambda_\epsilon(A + \gamma G) \cap B)/\epsilon^2$$

Lemma 3 shows that

$$\liminf_{\epsilon \to 0} \mathbb{E} f_\epsilon(G) \leq n/\gamma^2$$

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So by Lemma 1:

$$\mathbb{E}\, \pi \sum_{\lambda_i \in B} \kappa^2(\lambda_i) = \mathbb{E} \liminf_{\epsilon \to 0} f_\epsilon(G) \leq \frac{n}{\gamma^2} \cdot vol(B)$$

□

Recap of the Proof

Let $M = A + \gamma G$ and $B = D(0,3)$.

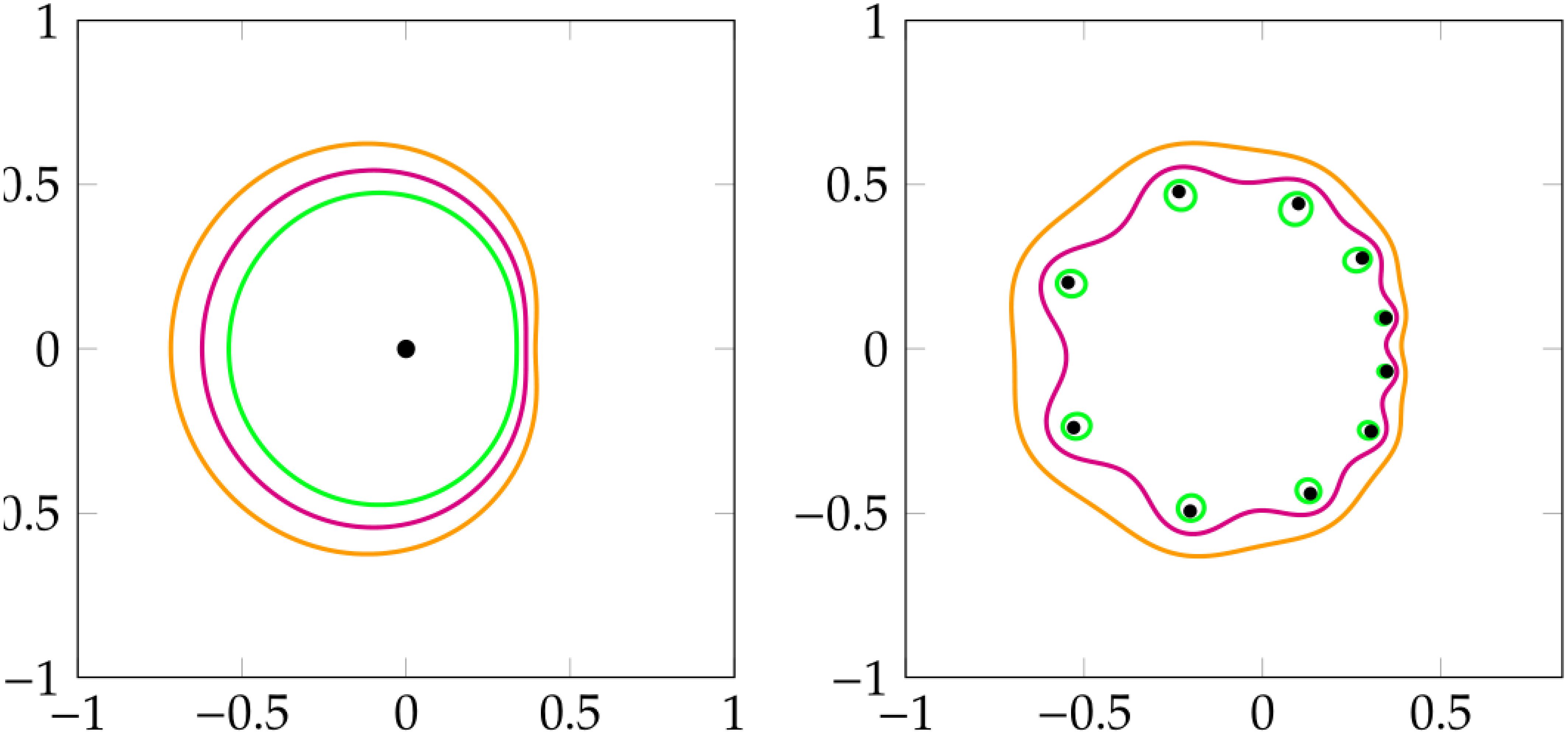
$$\mathbb{E} \sum_{\lambda_i \in B} \kappa(\lambda_i)^2 = \frac{1}{\pi} \cdot \mathbb{E} \liminf_{\epsilon \to 0} \frac{vol(\Lambda_\epsilon(M) \cap B)}{\epsilon^2}$$

$$\leq \frac{1}{\pi} \cdot \liminf_{\epsilon \to 0} \mathbb{E} \frac{vol(\Lambda_\epsilon(M) \cap B)}{\epsilon^2}$$

$$\leq \frac{9 \max_{z \in B} \mathbb{P}[z \in \Lambda_\epsilon(M)]}{\epsilon^2}$$

$$\leq 9n/\gamma^2$$

Phenomenon behind the result



Summary and Questions

Three related notions of spectral stability (κ_e, $\kappa(\lambda_i)$, Λ_ϵ)

Can control global quantities by local singular values $\sigma_n(z - M)$

Exploited invariance and anticoncentration of complex Gaussian

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Three related notions of spectral stability (κ_e, $\kappa(\lambda_i)$, Λ_ϵ)

Can control global quantities by local singular values $\sigma_n(z - M)$

Exploited invariance and anticoncentration of complex Gaussian

- Does a real Gaussian fail?
- Dimension dependence in Theorem A. Dimension free bound?
- Derandomization of the perturbation
- Non-gaussian perturbations