# Recent breakthroughs in sphere packing

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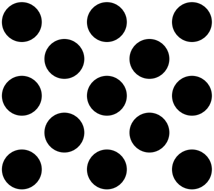
# Sphere packings

**Definition**

A sphere packing in $\mathbb{R}^n$ is a collection of spheres/balls of equal size which do not overlap (except for touching). The density of a sphere packing is the volume fraction of space occupied by the balls.



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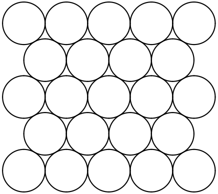
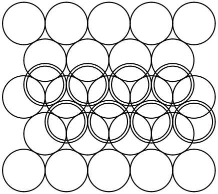
In dimension 2, the best possible is by using the hexagonal lattice. [Fejes Tóth 1940]



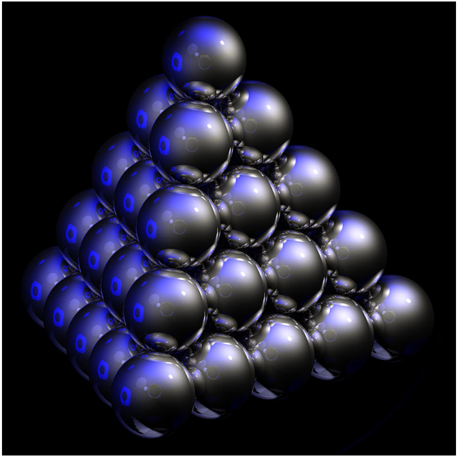
# Sphere packing problem II

In dimension 3, the best possible way is to stack layers of the solution in 2 dimensions. This is Kepler's conjecture, now a theorem of Hales.



There are infinitely (in fact, uncountably) many ways of doing this! These are the Barlow packings.

# Face centered cubic packing



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In very high dimensions (say $\geq 1000$) densest packings are likely to be close to disordered.

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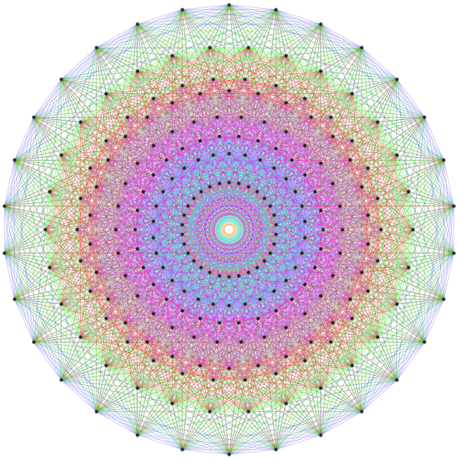
  - $E_6$ orthogonal complement of an $A_2$ in $E_8$.

# Projection of E8 root system



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My favorite: The lattice $\mathrm{II}_{25,1}$ is generated in $\mathbb{R}^{25,1}$ (which has the quadratic form $x_1^2 + \cdots + x_{25}^2 - x_{26}^2$) by vectors in $\mathbb{Z}^{26}$ or $(\mathbb{Z} + 1/2)^{26}$ with even coordinate sum.

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The Leech lattice is $w^{\perp}/\mathbb{Z}w$ with the induced quadratic form.

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The packing problem for lattices asks for the densest lattice(s) in $\mathbb{R}^n$ for every $n$. This is equivalent to the determination of the Hermite constant $\gamma_n$, which arises in the geometry of numbers. The known answers are:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| $\Lambda$ | $A_1$ | $A_2$ | $A_3$ | $D_4$ | $D_5$ | $E_6$ | $E_7$ | $E_8$ | Leech |
| due to | | Lagrange | Gauss | Korkine-Zolotareff | | Blichfeldt | | | Cohn-Kumar |

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1. linear programming bounds for packing
2. the theory of modular forms

# Linear programming bounds

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*Then the density of any sphere packing in $\mathbb{R}^n$ is bounded above by*

$$vol(B_n)(r/2)^n.$$

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Note that the constraints and objective function given are linear in $f$. Therefore this is a linear (convex) program.

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Now the LHS is $\leq f(0)$ while the sum in the RHS is $\geq \hat{f}(0) \geq 1$, yielding

$$\frac{1}{\mathrm{covol}(\Lambda)} \leq f(0)$$

multiplying by the volume of a ball of radius $1/2$ tells us that the density is at most $2^{-n} vol(B_n) f(0)$.

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$$f(x) = \sum_{i=0}^{N} c_i L_i(2\pi|x|^2) \exp(-\pi|x|^2)$$

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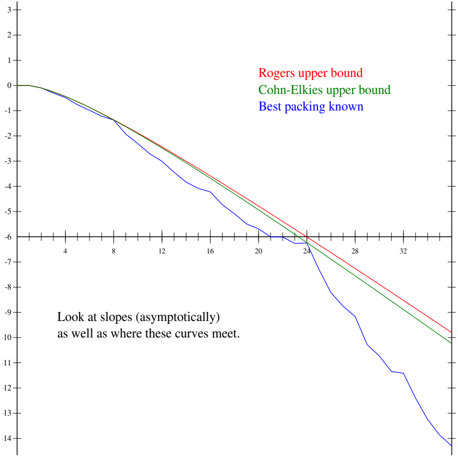
  where $c_i$ are the coefficients of the linear program, $L_i$ are the Laguerre polynomials (so $L_i$ times Gaussian is an eigenfunction for the Fourier transform).
- In dimensions 8 and 24 one can get upper bounds which are numerically very close to the lower bound coming from $E_8$ or Leech density.

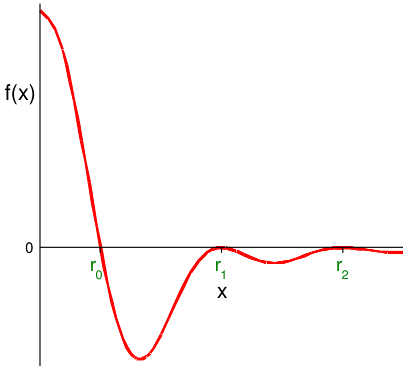
# LP bounds with dimension

Here is a plot of log$(\mathrm{density})$ vs. dimension.



# Desired functions

Let $\Lambda$ be $E_8$ or the Leech lattice, and $r_0, r_1, \ldots$ its nonzero vector lengths (square roots of the even natural numbers, except Leech skips 2). To have a tight upper bound that matches $\Lambda$, we need the function $f$ to look like this:



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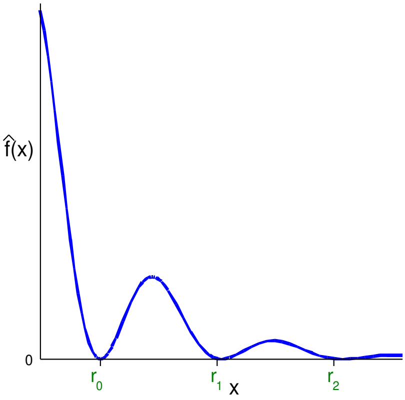
While $\hat{f}$ must look like this:



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We were stuck for more than a decade.

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Specifically, let $\mathrm{SL}_2(\mathbb{Z})$ denote all the integer two by two matrices of determinant 1.

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In fact the action factors through $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$, and this quotient group is generated by the images of

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

# Fundamental domain

The picture shows Dedekind's famous tesselation of the upper half plane. The union of a black and a white region makes a fundamental domain for the action of $SL_2(\mathbb{Z})$.

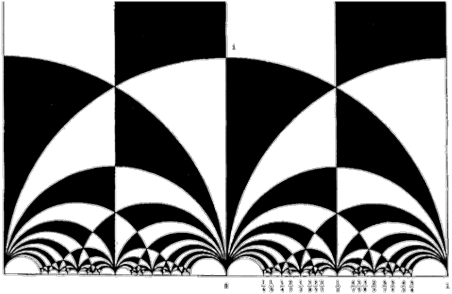

Image from the blog neverendingbooks.org, originally from John Stillwell's article "Modular miracles" in Amer. Math. Monthly.

# Modular curves

The quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ can be identified with the Riemann sphere $\mathbb{CP}^1$ minus a point. Compactifying the quotient by adding this cusp gives an algebraic curve (namely $\mathbb{CP}^1$).

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The principal congruence subgroup of level $N$ is the subgroup $\Gamma(N)$ of all the elements of $\mathrm{SL}_2(\mathbb{Z})$ congruent to the identity modulo $N$. We say $\Gamma$ is a congruence subgroup if it contains some $\Gamma(N)$. Again the quotient is a complex algebraic curve; we can compactify it by adding finitely many cusps, which correpond to the elements of $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$.

# Modular forms

The first condition for a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ to be a modular form for $\Gamma$ of weight $k$ is

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

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Now, for some $N$ the matrix

$$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$$

lies in the congruence subgroup, so we must have $f(z+N) = f(z)$.

# Growth condition

So if $q = \exp(2\pi i z)$ then we can write $f$ as a function of $q^{1/N}$.

The second condition for a modular form says that near $\infty$, there is a power series expansion

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Similarly for all the (finitely many) cusps. Defining the slash operator for $g \in \mathrm{SL}_2(\mathbb{Z})$ as above by

$$(f|_k g)(z) = (cz + d)^{-k} f(gz),$$

all these $f|_k g$ must have holomorphic power series expansion at $\infty$.

# Growth condition

So if $q = \exp(2\pi i z)$ then we can write $f$ as a function of $q^{1/N}$.

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all these $f|_k g$ must have holomorphic power series expansion at $\infty$.

If it's only a Laurent series, i.e., there are (finitely many) negative powers of q, we say that $f$ is a weakly holomorphic modular form.

# Examples

How do we find actual examples of modular forms?

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The first way is to take simple examples of a "well-behaved" holomorphic function and symmetrize (recalling that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$):

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$$G_k(z) = \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(az+b)^k}.$$

For even $k \geq 4$, the sum converges absolutely and we get a non-zero modular form of weight $k$. These are called Eisenstein series.

# Eisenstein series

The normalized versions are

$$E_4 = 1 + 240 \sum \sigma_3(n) q^n$$
$$E_6 = 1 - 504 \sum \sigma_5(n) q^n$$

Here $\sigma_k(n) = \sum_{d|n, d>0} d^k$.

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These two in fact generate the algebra of modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Another beautiful example is the modular discriminant of weight 12

$$\Delta = (E_4^3 - E_6^2)/1728 = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

# Theta functions

Another source of modular forms is theta functions of lattices:

If $\Lambda$ is an integral lattice (i.e. all inner products between vectors in the lattice are integers) of dimension $d$ then

$$\Theta_\Lambda(q) = \sum_{v \in \Lambda} q^{\langle v,v \rangle/2} = \sum_{n \geq 0} N_n(\Lambda) q^{n/2}$$

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**Example**

The theta function of $E_8$ is the Eisenstein series $E_4$!

# Theta functions II

There are also classical theta functions studied by Jacobi, of which we will need:

$$\Theta_{00}(z) := \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 z)$$

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Let $U = \Theta_{00}^4$, $V = \Theta_{10}^4$, $W = \Theta_{01}^4$. These are modular forms of weight 2 for the congruence subgroup $\Gamma(2)$.

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Let $U = \Theta_{00}^4$, $V = \Theta_{10}^4$, $W = \Theta_{01}^4$. These are modular forms of weight 2 for the congruence subgroup $\Gamma(2)$. They are related by $U = V + W$.

# L-functions

Usually, from a modular form we make an $L$-function by taking a Mellin transform:

$$L(f,s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(it) t^s \frac{dt}{t}$$

which works for $\Re(s)$ large enough.

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These $L$-functions are a cornerstone of much of modern number theory.

For instance, Wiles's proof of FLT relies on showing the $L$-function of a specific kind of elliptic curve is the same as that of a modular form.

# Quasimodular forms

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$$G_2(z) = \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2}$$

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and this double sum converges. Normalizing we have

$$E_2 = 1 - 24 \sum_{n \geq 0} \sigma_1(n) q^n.$$

The only problem is that $E_2$ is not a genuine modular form:

$$E_2(-1/z) = z^2 E_2(z) - \frac{6i}{\pi} z.$$

# Quasimodular forms II

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It can also be obtained by differentiating modular forms. For

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In general differentiating a weight $k$ modular forms of weight $\ell$ times yields a polynomial in $E_2$ of degree $\ell$, and the resulting quasimodular form has weight $k + 2\ell$. We call $\ell$ the depth of the quasimodular form.

# Even eigenfunction

The magic functions for sphere packing arise as (Laplace) transforms of weakly holomorphic modular or quasi-modular forms.

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and for $r > \sqrt{2}$, define

$$a(r) = -4\sin(\pi r^2/2)^2 \int_0^{i\infty} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} dz.$$

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We can extend give an alternative expression for the integral which extends the domain of definition to $r > 0$.

# Even eigenfunction II

Note that:

- $\phi_0(-1/(it))t^2 = O(\exp(2\pi t))$ as $t \to \infty$. So the integral has a term proportional to

$$\int_0^\infty \exp(-\pi(r^2-2)t)dt = \frac{1}{\pi(r^2-2)}$$

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- The quasi-modular property of $\phi_0$ can be used to show that $a(r)$ is an even eigenfunction: the Fourier transform replaces $e^{\pi i r^2 z}$ by $z^{-4}e^{\pi i r^2(-1/z)}$ and then we can use transformation properties under $z \to -1/z$.

# Even eigenfunction III

Write

$$-4\sin^2(\pi r^2/2) = -2(1-\cos(\pi r^2)) = \exp(\pi i r^2) + \exp(-\pi i r^2) - 2.$$

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$$a(r) = \int_0^{i\infty} \phi_0(-1/z) z^2 \left( e^{\pi i r^2 (z+1)} + e^{\pi i r^2 (z-1)} - 2e^{\pi i r^2 z} \right) dz$$

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&= \int_0^{i\infty} \phi_0(-1/z) z^2 e^{\pi i r^2 (z+1)} dz + \int_0^{i\infty} \phi_0(-1/z) z^2 e^{\pi i r^2 (z-1)} dz \\
&\quad - 2 \int_0^{i\infty} \phi_0(-1/z) z^2 e^{\pi i r^2} dz \\
&= \int_1^{i\infty+1} \phi_0\left(\frac{-1}{u-1}\right) (u-1)^2 e^{\pi i r^2 u} du + \int_{-1}^{i\infty-1} \phi_0\left(\frac{-1}{u+1}\right) (u+1)^2 e^{\pi i r^2 u} du \\
&\quad - 2 \int_0^{i\infty} \phi_0(-1/z) z^2 e^{\pi i r^2 z} du
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# Even eigenfunction IV

We can shift the contour at infinity, and break up the path.

$$
\begin{aligned}
a(r) = &\int_1^i \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i r^2 z}\,dz + \int_i^{i\infty} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i r^2 z}\,dz \\
&+ \int_{-1}^i \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z}\,dz + \int_i^{i\infty} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z}\,dz \\
&- 2\int_0^i \phi_0(-1/z) z^2 e^{\pi i r^2}\,dz - 2\int_i^{i\infty} \phi_0(-1/z) z^2 e^{\pi i r^2}\,dz
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We will combine the second, fourth and sixth integrals. Note that

$$z^2\phi_0(-1/z) = z^2\phi_0(z) + z\phi_{-2}(z) + \phi_{-4}(z)$$

where $\phi_0, \phi_{-2}, \phi_{-4}$ are quasimodular forms of depth $2, 1, 0$ and weight $0, -2, -4$ respectively. In any case, they are all invariant under $T$.

# Even eigenfunction V

Therefore, the second difference operator just acts on the multipliers on $z^2, z, 1$, yielding

$$\phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 + \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 - \phi_0\left(\frac{-1}{z}\right)z^2$$
$$= \phi_0(z)\big((z+1)^2 + (z-1)^2 - 2z^2\big) = 2\phi_0(z).$$

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Therefore

$$a(r) = \int_1^i \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i r^2 z}\,dz + \int_{-1}^i \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z}\,dz$$
$$- 2\int_0^i \phi_0(-1/z) z^2 e^{\pi i r^2 z}\,dz + 2\int_i^{i\infty} 2\phi_0(z) e^{\pi i r^2 z}\,dz.$$

# Fourier transform

We have

$$\widehat{a}(r) = \int_1^i \phi_0\left(\tfrac{-1}{z-1}\right)\frac{(z-1)^2}{z^4}e^{\pi i r^2\left(\frac{-1}{z}\right)}dz + \int_{-1}^i \phi_0\left(\tfrac{-1}{z+1}\right)\frac{(z+1)^2}{z^4}e^{\pi i r^2\left(\frac{-1}{z}\right)}dz$$
$$-2\int_0^i \phi_0(-1/z)z^2z^{-4}e^{\pi i r^2(-1/z)}dz - 2\int_i^{i\infty} 2\phi_0(z)z^{-4}e^{\pi i r^2(-1/z)}dz$$

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using the change of variable $z = -1/w$, $dz = 1/w^2dw$, and the $T$-invariance of $\phi_0$.

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using the change of variable $z = -1/w$, $dz = 1/w^2 dw$, and the $T$-invariance of $\phi_0$.

So we have created a $+1$-eigenfunction for the Fourier transform.

# Odd eigenfunction

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$$\psi = \frac{2W^3(5UV + 2W^2)}{\Delta}.$$

It is a weakly holomorphic modular form of weight $-2$ for the congruence subgroup $\Gamma_0(2)$.

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We can similarly show that $b(r)$ is an odd eigenfunction for the Fourier transform, and has a single root at $r = \sqrt{2}$ and double roots at other $\sqrt{2n}$.

# Odd eigenfunction II

Write $\psi_T = \psi|_{-2}T$ and $\psi_S = \psi|_{-2}S$. Then it is easy to verify that $\psi_S + \psi_T = \psi$, from which it follows that $\psi_T|_{-2}S = -\psi_T$. Also, $\psi_S|_{-2}S = \psi$ and finally $\psi|_{-2}T^{-1} = \psi_T$ since $T^{-2} \in \Gamma(2)$.

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We rewrite the integral as before

$$b(r) = \int_0^{i\infty} \psi(z) e^{\pi i r^2 (z+1)} dz + \int_0^{i\infty} \psi(z) e^{\pi i r^2 (z-1)} dz - 2 \int_0^{i\infty} \psi(z) e^{\pi i r^2 z} dz$$

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&\quad - 2 \int_0^{i\infty} \psi(z) e^{\pi i r^2 z} dz \\
&= \int_1^{i\infty} \psi(z-1) e^{\pi i r^2 z} dz + \int_{-1}^{i\infty} \psi(z+1) e^{\pi i r^2 z} dz \\
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\begin{aligned}
b(r) &= \int_0^{i\infty} \psi(z) e^{\pi i r^2 (z+1)} dz + \int_0^{i\infty} \psi(z) e^{\pi i r^2 (z-1)} dz \\
&\quad - 2\int_0^{i\infty} \psi(z) e^{\pi i r^2 z} dz \\
&= \int_1^{i\infty} \psi(z-1) e^{\pi i r^2 z} dz + \int_{-1}^{i\infty} \psi(z+1) e^{\pi i r^2 z} dz \\
&\quad - 2\int_0^{i\infty} \psi(z) e^{\pi i r^2 z} dz \\
&= \int_1^{i\infty} \psi_T(z) e^{\pi i r^2 z} dz + \int_{-1}^{i\infty} \psi_T(z) e^{\pi i r^2 z} dz - 2\int_0^{i\infty} \psi(z) e^{\pi i r^2 z} dz
\end{aligned}
$$

# Odd eigenfunction III

$$b(r) = \int_1^i \psi_T(z) e^{\pi i r^2 z} dz + \int_{-1}^i \psi_T(z) e^{\pi i r^2 z} dz - 2 \int_0^i \psi(z) e^{\pi i r^2 z} dz + 2 \int_i^{i\infty} (\psi_T(z) - \psi(z)) e^{\pi i r^2 z} dz$$

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$$
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&\quad + 2\int_i^{i\infty} (\psi_T(z) - \psi(z)) e^{\pi i r^2 z}\,dz \\
&= \int_1^i \psi_T(z) e^{\pi i r^2 z}\,dz + \int_{-1}^i \psi_T(z) e^{\pi i r^2 z}\,dz - 2\int_0^i \psi(z) e^{\pi i r^2 z}\,dz \\
&\quad - 2\int_i^{i\infty} \psi_S(z) e^{\pi i r^2 z}\,dz.
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This extends the domain of definition to $r > 0$. Note that $\psi(it) = O(e^{2\pi t})$ as $t \to \infty$ gives a pole at $r = \sqrt{2}$ for the integral, just as in the even case.

To check that we have an odd eigenfunction, we compute

# Odd eigenfunction IV

$$\widehat{b}(r) = \int_1^i \psi_T(z) z^{-4} e^{\pi i r^2(-1/z)} dz + \int_{-1}^i \psi_T(z) z^{-4} e^{\pi i r^2(-1/z)} dz$$
$$- 2\int_0^i \psi(z) z^{-4} e^{\pi i r^2(-1/z)} dz - 2\int_i^{i\infty} \psi_S(z) z^{-4} e^{\pi i r^2(-1/z)} dz$$

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# Odd eigenfunction V

So

$$\begin{aligned}\widehat{b}(r) &= -\int_{-1}^{i} \psi_T(w)e^{\pi i r^2 w}\,dw - \int_{1}^{i} \psi_T(w)e^{\pi i r^2 w}\,dw \\ &\quad + 2\int_{i}^{\infty} \psi_S(w)e^{\pi i r^2 w}\,dw + 2\int_{0}^{i} \psi(w)e^{\pi i r^2 w}\,dw \\ &= -b(r)\end{aligned}$$

where we used $\psi_{TS} = -\psi_T$.

# Putting everything together

Now, we can take a linear combination of $a(r)$ and $b(r)$ to make $f$ such that $f$ and $\hat{f}$ have the desired properties (for instance, to make $\hat{f}$ vanish to order 2 at $\sqrt{2}$.

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At the moment, this last verification of the required inequalities needs a computer-assisted proof.

# Leech lattice

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For the even eigenfunction, the integrand has the weakly holomorphic quasimodular form

$$\phi = \frac{(25E_4^4 - 49E_6^2E_4) + 48E_6E_4^2E_2 + (-49E_4^3 + 25E_6^2)E_2^2}{\Delta^2}.$$

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For the odd eigenfunction, the integrand has the weakly holomorphic modular form for $\Gamma(2)$

$$\psi = \frac{W^5(7UV + 2W^2)}{\Delta^2}.$$

# Beyond sphere packing in 8 and 24 dimensions

One big open problem is to find magic functions for dimension 2 (even though we know the $A_2$ lattice gives the densest sphere packing, by a relatively elementary argument).

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One big open problem is to find magic functions for dimension 2 (even though we know the $A_2$ lattice gives the densest sphere packing, by a relatively elementary argument).

In other dimensions, we do not expect this technique to give sharp bounds, but it may yield better upper bounds for sphere packing than the current records.

We have since also worked on a wide generalization of the sphere packing problem to energy minimization, and have proved that $E_8$ and the Leech lattice are universally optimal for Gaussian (and therefore inverse power law) potential functions in their respective dimensions, via sharp LP bounds for energy.

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# Thank you!