The Jensen-Pólya Program for the Riemann Hypothesis and Related Problems

Ken Ono (U of Virginia)

Riemann's zeta-function

Definition (Riemann)

For $\mathrm{Re}(s) > 1$, define the **zeta-function** by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

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1. *The function $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ (apart from a simple pole at $s = 1$ with residue 1).*
2. *We have the* **functional equation**

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \cdot \zeta(1-s).$$

Hilbert's 8th Problem

Conjecture (Riemann Hypothesis)

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"Without doubt, it would be desirable to have a rigorous proof of this proposition; however, I have left this research...because it appears to be unnecessary for the immediate goal of my study...."

Bernhard Riemann (1859)

Important Remarks

Fact (Riemann's Motivation)

Proposed RH because of Gauss' Conjecture that $\pi(X) \sim \frac{X}{\log X}$.

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1. *The first "gazillion" zeros satisfy RH (van de Lune, Odlyzko).*
2. $> 41\%$ *of zeros satisfy RH (Selberg, Levinson, Conrey,...).*

JENSEN-PÓLYA PROGRAM

J. W. L. Jensen
(1859–1925)

George Pólya
(1887–1985)

Jensen-Pólya Program

Definition

The **Riemann Xi-function** is the entire function

$$\Xi(z) := \frac{1}{2}\left(-z^2 - \frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right).$$

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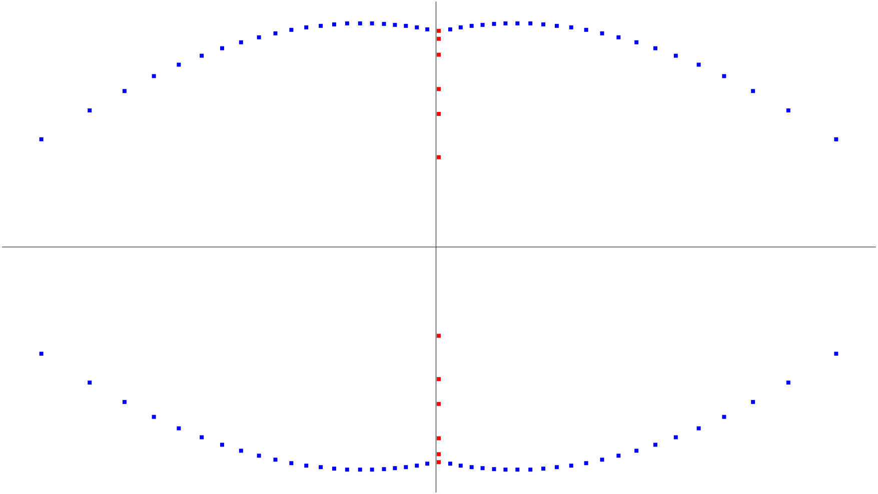
Remark

RH is true $\iff$ *all of the zeros of* $\Xi(z)$ *are purely real.*

Roots of Deg 100 Taylor Poly for $\Xi\left(\frac{1}{2}+z\right)$



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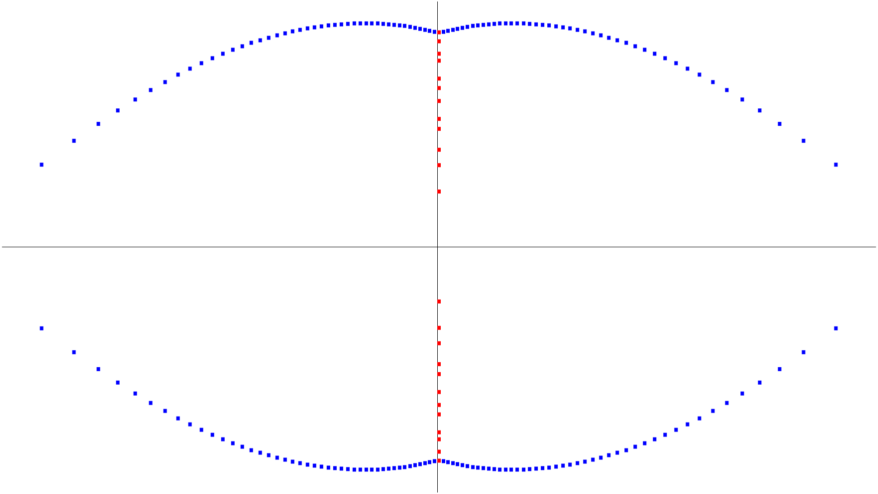
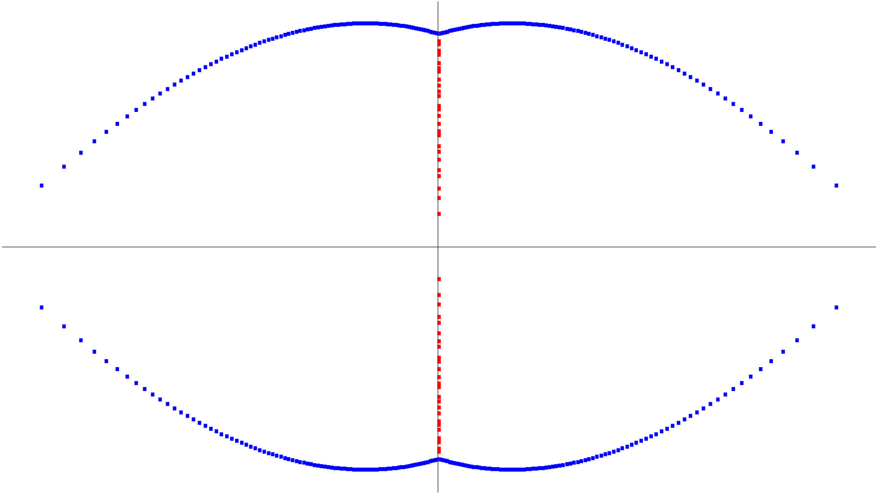


Roots of Deg 400 Taylor Poly for $\Xi\left(\frac{1}{2}+z\right)$



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- As $d \to +\infty$ the spurious points become more prevalent.

JENSEN POLYNOMIALS

DEFINITION (JENSEN)

The **degree** d **and shift** n **Jensen polynomial** for an arithmetic function $a : \mathbb{N} \mapsto \mathbb{R}$ is

$$J_a^{d,n}(X) := \sum_{j=0}^{d} a(n+j) \binom{d}{j} X^j$$
$$= a(n+d)X^d + a(n+d-1)dX^{d-1} + \cdots + a(n).$$

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Definition

A polynomial $f \in \mathbb{R}[X]$ is **hyperbolic** if all of its roots are real.

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$$\left(-1+4z^2\right)\Lambda\left(\frac{1}{2}+z\right) = \sum_{n=0}^{\infty} \frac{\gamma(n)}{n!} \cdot z^{2n}.$$

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RH **is equivalent to the hyperbolicity** *of all of the* $J_\gamma^{d,n}(X)$.

What was known?

The hyperbolicity for all n is known for $d \leq 3$ by work of Csordas, Norfolk and Varga, and Dimitrov and Lucas.

New Theorems

"Theorem 1" (Griffin, O, Rolen, Zagier)

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Theorem (O+)

If $d > 1$ and $n \gg e^{8d/9}$, then $J_{\gamma}^{d,n}(X)$ is hyperbolic.

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2. *We* **"locate"** *the real zeros of the* $J_{\gamma}^{d,n}(X)$.
3. *Wagner has extended the 1st theorem to other L-functions.*

Hermite Polynomials

Definition

The (modified) **Hermite polynomials**

$$\{H_d(X) \ : \ d \geq 0\}$$

are the orthogonal polynomials with respect to $\mu(X) := e^{-\frac{X^2}{4}}$.

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Example (The first few Hermite polynomials)

$$H_0(X) = 1$$
$$H_1(X) = X$$
$$H_2(X) = X^2 - 2$$
$$H_3(X) = X^3 - 6X$$

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1. *Each* $H_d(X)$ *is* **hyperbolic** *with* d **distinct** *roots.*
2. *If* S_d *denotes the "suitably normalized" zeros of* $H_d(X)$*, then*

$$S_d \longrightarrow \text{Wigner's Semicircle Law.}$$

RH Criterion and Hermite Polynomials

Theorem 1 (Griffin, O, Rolen, Zagier)

The **renormalized** *Jensen polynomials* $\widehat{J}_{\gamma}^{d,n}(X)$ *satisfy*

$$\lim_{n \to +\infty} \widehat{J}_{\gamma}^{d,n}(X) = H_d(X).$$

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For each d at most finitely many $J_{\gamma}^{d,n}(X)$ **are not** *hyperbolic.*

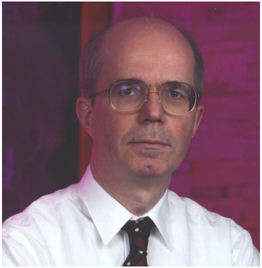
Degree 3 Normalized Jensen polynomials

n	$\widehat{J}_{\gamma}^{3,n}(X)$
100	$\approx 0.9769X^3 + 0.7570X^2 - 5.8690X - 1.2661$
200	$\approx 0.9872X^3 + 0.5625X^2 - 5.9153X - 0.9159$
300	$\approx 0.9911X^3 + 0.4705X^2 - 5.9374X - 0.7580$
400	$\approx 0.9931X^3 + 0.4136X^2 - 5.9501X - 0.6623$
$\vdots$	$\vdots$
10^8	$\approx 0.9999X^3 + 0.0009X^2 - 5.9999X - 0.0014$
$\vdots$	$\vdots$
∞	$H_3(X) = X^3 - 6X$

Random Matrix Model Predictions


Freeman Dyson


Hugh Montgomery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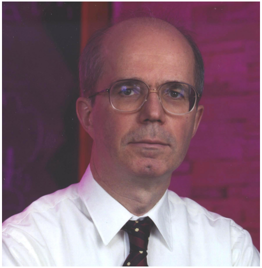

Andrew Odlyzko

Gaussian Unitary Ensemble (GUE) (1970s)

The nontrivial zeros of $\zeta(s)$ appear to be "distributed like" the eigenvalues of random Hermitian matrices.

Relation to our work

"Theorem" (Griffin, O, Rolen, Zagier)

GUE holds for Riemann's $\zeta(s)$ in **derivative aspect**.

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$$\lim_{n\to+\infty} \widehat{J}_{\gamma}^{d,n}(X) = H_d(X).$$

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3. *For fixed d, we proved that*

$$\lim_{n \to +\infty} \widehat{J}_{\gamma}^{d,n}(X) = H_d(X).$$

4. *The zeros of the $\{H_d(X)\}$ and the eigenvalues in GUE both satisfy Wigner's Semicircle Distribution.* □

Computing derivatives Is not Easy

Theorem (Pustylnikov (2001), Coffey (2009))

As $n \to +\infty$, we have

$$\xi^{(2n)}(1/2) = \frac{(2n)(2n-1)(2n-2)^{\frac{-1}{4}}}{2^{2n-2}\ln(2n-2)^{\frac{1}{4}}}\left[\ln\left(\frac{2n-2}{\pi}\right) - \ln\ln\left(\frac{2n-2}{\pi}\right) + o(1)\right]^{2n-\frac{3}{2}} \times \exp\left(-\frac{2n-2}{\ln(2n-2)}\right).$$

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Remarks

1. *Derivatives essentially drop to 0 for "small" n before exhibiting* **exponential growth.**
2. *This* **is insufficient** *for approximating* $J_{\gamma}^{d,n}(X)$.

First 10 Taylor coefficients of $\Xi(x)$

m	$\hat{b}_m$
0	6.214 009 727 353 926 (–2)
1	7.178 732 598 482 949 (–4)
2	2.314 725 338 818 463 (–5)
3	1.170 499 895 698 397 (–6)
4	7.859 696 022 958 770 (–8)
5	6.474 442 660 924 152 (–9)
6	6.248 509 280 628 118 (–10)
7	6.857 113 566 031 334 (–11)
8	8.379 562 856 498 463 (–12)
9	1.122 895 900 525 652 (–12)
10	1.630 766 572 462 173 (–13)

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3. *Let* $L = L(n) \approx \log\left(\frac{n}{\log n}\right)$ *be the unique positive solution of the equation* $n = L \cdot (\pi e^L + \frac{3}{4})$.

Arbitrary precision asymptotics

Theorem (Griffin, O, Rolen, Zagier)

To **all orders**, *as* $n \to +\infty$, *there are* $b_k \in \mathbb{Q}(L)$ *such that*

$$F(n) \sim \sqrt{2\pi} \frac{L^{n+1}}{\sqrt{(1+L)n - \frac{3}{4}L^2}} e^{L/4 - n/L + 3/4} \left(1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots\right),$$

where $b_1 = \frac{2L^4+9L^3+16L^2+6L+2}{24\,(L+1)^3}$.

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Remarks

1. *Using two terms (i.e. b_1) suffices for our RH application.*
2. **Analysis** + **Computer** $\implies$ *hyperbolicity for $d \leq 10^{20}$.*

EXAMPLE: $\widehat{\gamma}(n) :=$ TWO-TERM APPROXIMATION

n	$\widehat{\gamma}(n)$	$\gamma(n)$	$\gamma(n)/\widehat{\gamma}(n)$
10	$\approx 1.6313374394 \times 10^{-17}$	$\approx 1.6323380490 \times 10^{-17}$	≈ 1.000613367
100	$\approx 6.5776471904 \times 10^{-205}$	$\approx 6.5777263785 \times 10^{-205}$	≈ 1.000012038
1000	$\approx 3.8760333086 \times 10^{-2567}$	$\approx 3.8760340890 \times 10^{-2567}$	≈ 1.000000201
10000	$\approx 3.5219798669 \times 10^{-32265}$	$\approx 3.5219798773 \times 10^{-32265}$	≈ 1.000000002
100000	$\approx 6.3953905598 \times 10^{-397097}$	$\approx 6.3953905601 \times 10^{-397097}$	≈ 1.000000000

How do these asymptotics imply Theorem 1?

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Theorem 1 is an example of a **general phenomenon**!

Hyperbolic Polynomials in Mathematics

Remark

Hyperbolicity of "generating polynomials" is studied in enumerative combinatorics in connection with **log-concavity**

$$a(n)^2 \geq a(n-1)a(n+1).$$

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- *Group theory (lattice subgroup enumeration)*
- *Graph theory*
- *Symmetric functions*
- *Additive number theory (partitions)*
- ...

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for each j for real sequences $\{A(n)\}$ and $\{\delta(n)\} \to 0$.

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$$\log\left(\frac{a(n+j)}{a(n)}\right)$$

$$= A(n)j - \delta(n)^2 j^2 + \sum_{i=0}^{d} \mathrm{o}_{i,d}(\delta(n)^i) j^i + \mathrm{O}_d\left(\delta(n)^{d+1}\right).$$

General Theorem

Definition

If $a(n)$ has appropriate growth, then the **renormalized Jensen polynomials** are defined by

$$\widehat{J}_a^{d,n}(X) := \frac{1}{a(n)\cdot\delta(n)^d}\cdot J_a^{d,n}\left(\frac{\delta(n)X-1}{\exp(A(n))}\right).$$

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$$p(n) := \#\text{partitions of size } n.$$

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A **partition** is any nonincreasing sequence of integers.

$$p(n) := \#\text{partitions of size } n.$$

Example

We have that $p(4) = 5$ because the partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Log concavity of $p(n)$

Example

The roots of the quadratic $J_p^{2,n}(X)$ are

$$\frac{-p(n+1) \pm \sqrt{p(n+1)^2 - p(n)p(n+2)}}{p(n+2)}.$$

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Theorem (Nicolas (1978), DeSalvo and Pak (2013))

If $n \geq 25$, *then* $J_p^{2,n}(X)$ *is hyperbolic.*

Chen's Conjecture

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Table 1. Conjectured minimal values of $N(d)$

d	1	2	3	4	5	6	7	8	9
$N(d)$	1	25	94	206	381	610	908	1269	1701

Our result

Theorem 2 (Griffin, O, Rolen, Zagier)

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1. *The proof can be refined case-by-case to prove the minimality of the claimed $N(d)$ (Larson, Wagner).*

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Remarks

1. *The proof can be refined case-by-case to prove the minimality of the claimed $N(d)$ (Larson, Wagner).*
2. *This is a consequence of the* **General Theorem**.

Modular forms

Definition

A **weight** k **weakly holomorphic modular form** is a function f on $\mathbb{H}$ satisfying:

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$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

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Example (Partition Generating Function)

We have the weight $-1/2$ modular form

$$f(\tau) = \sum_{n=0}^{\infty} p(n) e^{2\pi i \tau (n - \frac{1}{24})}.$$

Jensen polynomials for modular forms

Theorem 3 (Griffin, O, Rolen, Zagier)

Let f be a weakly holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ with real coefficients and a pole at $i\infty$. Then for each degree $d \geq 1$

$$\lim_{n \to +\infty} \widehat{J}_{a_f}^{d,n}(X) = H_d(X).$$

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Sketch of Proof. Sufficient asymptotics are known for $a_f(n)$ in terms of Kloosterman sums and Bessel functions.

Natural Questions

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What is special about the Hermite polynomials?

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Is there an even more general theorem?

Hermite Polynomial Generating Function

Lemma (Generating Function)

We have that

$$e^{-t^2+Xt} =: \sum_{d=0}^{\infty} H_d(X) \cdot \frac{t^d}{d!} = 1 + X \cdot t + (X^2 - 2) \cdot \frac{t^2}{2} + \ldots$$

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Remark

The rough idea of the proof is to show for large fixed n that

$$\sum_{d=0}^{\infty} \widehat{J}_a^{d,n}(X) \cdot \frac{t^d}{d!} \approx e^{-t^2+Xt} = e^{-t^2} \cdot e^{Xt}.$$

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How does the **shape** *of $F(t)$ impact "limiting polynomials"?*

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Some Remarks

Remark (Limit Polynomials)

If $a : \mathbb{N} \mapsto \mathbb{R}$ *is appropriate for* $F(t)$*, then*

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(3) $F(t) = e^{-t^2} \implies \widehat{H}_d(X) = H_d(X)$ **Hermite poly**.

Loose End

Theorem (O+)

Height T RH $\Longrightarrow$ hyperbolicity of $J^{d,n}(X)$ for all n if $d \leq T^2$.

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- Truth of RH for low height interfaces well with differentiation.

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If $d \geq 1$ and $n \gg e^{8d/9}$, then $J_{\gamma}^{d,n}(X)$ is hyperbolic.

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Polya criterion for polys given as a Hermite decomposition.

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A sequence with appropriate growth for $F(t) = e^{-t^2}$ **has type** $Z : \mathbb{N} \to \mathbb{R}^+$ if $J_a^{d,n}(X)$ is hyperbolic for $n \geq Z(d)$.

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1. *RH is equivalent to $\gamma(n)$ having type $Z = 0$.*
2. *For $\gamma(n)$ we have proved that $Z(d) = O(e^{8d/9})$.*
3. **Have heuristics** *for $Z(d)$ for modular form coefficients.*

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Evidence

If we let $\widehat{Z}(d) := 10d^2 \log d$, *then we have*

d	$N(d)$	$\widehat{Z}(d)$	$N(d)/\widehat{Z}(d)$
1	1	≈ 1	≈ 1.00
2	25	≈ 27.72	≈ 0.90
4	206	≈ 221.80	≈ 0.93
8	1269	≈ 1330.84	≈ 0.95
16	6917	≈ 7097.82	≈ 0.97
32	35627	≈ 35489.13	≈ 1.00

Our Results

General Theorem (Griffin, O, Rolen, Zagier)

If $a(n)$ has **appropriate growth**, *then for $d \geq 1$ we have*

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Applications

Hermite Distributions

1. Jensen-Pólya criterion for RH **whenever** $n \gg e^{8d/9}$.
2. Jensen-Pólya criterion for RH **for all** n if $1 \leq d \leq 10^{20}$.
3. Height T RH $\Rightarrow$ Jensen-Pólya criterion **for all** n if $d \leq T^2$.
4. The **derivative aspect** GUE model for Riemann's $\Xi(x)$.
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$+$ general theory including Bernoulli and Eulerian distributions.