8. Martingales and Brownian motion.

37. Problems 2.15, 2.16, 2.17 and 2.18 in [**MP**].

38. Let $f(t) = 1 + \alpha\sqrt{t}$ where $\alpha > 0$ is fixed. Let $\tau_\alpha = \inf\{t : W_t = |f(t)|\}$. Show that τ_α is finite almost surely, but $\mathbf{E}[\tau_\alpha]$ is finite for $\alpha < 1$ and infinite for $\alpha > 1$.

39. Let $\mathbf{W}$ be a d-dimensional Brownian motion. Let $A_r = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = r\}$ and let τ_r be the first hitting time of the set A_r.

1. Let $0 < r < t < R$. For $\|x\| = t$, show that

$$\mathbf{P}_x(\tau_R < \tau_r) = \begin{cases} \frac{\log t - \log r}{\log R - \log r} & \text{if } d = 2 \\ \frac{t^{-d+2} - r^{-d+2}}{R^{-d+2} - r^{-d+2}} & \text{if } d \geq 3. \end{cases}$$

2. Deduce that Brownian motion is transient in $d \geq 3$ and is neighbourhood-recurrent in $d = 2$.

[**Hint:** For the first part, find a martingale. Look at Corollary 2.53 in [**MP**]].

40. Exponential martingales can sometimes be used to compute the distribution of stopping times. Here consider one-dimensional Brownian motion and let $a < 0 < b$ and compute the Laplace transforms of the following stopping times.

1. Show that $\mathbf{E}[e^{-\lambda\tau_b}] = e^{-\lambda\sqrt{2b}}$ for $\lambda > 0$.
2. Show that $\mathbf{E}[e^{\lambda\tau_{a,b}}] = \frac{\sinh\lambda b - \sinh\lambda a}{\sinh\lambda(b-a)}$ for any $\lambda \in \mathbb{R}$. Either from this, or directly, derive the first two moments of $\tau_{a,b}$.