3. Brownian motion. Let $C(I)$ be the space of continuous functions on I with the sup-norm and the corresponding Borel sigma-algebra. If $W = (W_t)_{t\in I}$ is the standard Brownian motion as defined in Problem 12, it is easy to check that W is a $C(I)$-valued random variable. The distribution of W is called *Wiener measure*, a Borel probability measure on $C(I)$.

Some generalities. Two $C(I)$-valued random variables X,Y have the same distribution if and only if they have the same finite dimensional distributions. This shows that SBM is well-defined as a $C[0,1]$-valued random variable and that the properties (a),(b),(c) in Problem 13 constitute an equivalent definition of SBM. Also, to check independence of two $C(I)$-valued random variables X and Y, it is enough to check that $(X(t_1),\dots,X(t_n))$ is independent of $(Y(s_1),\dots,Y(s_m))$ for any $t_j,s_i \in I$. **Advisory:** In the problems below, it is better to use the defining properties of Brownian motion rather than any particular construction of it.

14 (*Regularity of Wiener measure)***.** For any $\varepsilon > 0$, there exists a compact set $K \subseteq C(I)$ such that $\mathbf{P}(W \in K) \geq 1-\varepsilon$. Further, any open set of $C_0(I) = \{f \in C(I) : f(0) = 0\}$ has positive Wiener measure.

15. Let $0 = t_0 < t_1 < \ldots < t_m < 1$ and $x_1,\dots,x_m \in \mathbb{R}$ and $x_0 = 0$. Let W be standard BM. For $0 \leq k \leq m$ define,

$$B_k(s) := \frac{W(t_k(1-s)+t_{k+1}s) - (x_k(1-s)+x_{k+1}s)}{\sqrt{t_{k+1}-t_k}}, \quad \text{for } s \in I.$$

Then, conditional on $W(t_1) = x_1,\dots,W(t_m) = x_m$, the random functions $B_1,\dots,B_m$ are independent standard Brownian bridges.

16 (**Exercise 1.9** in **[MP]**)**.** If $\alpha > \frac{1}{2}$, then standard Brownian motion is nowhere Hölder-α continuous. Here Hölder-α continuity of f at a point t_0 means that $\limsup\limits_{h\to 0} \frac{|f(t_0+h)-f(t_0)|}{h^\alpha} < \infty$. [**Remark:** Observe that this proof does not work for Hölder-$1/2$ points].

17 (**Exercise 1.12** in **[MP]**)**.** In addition, for $\alpha > \frac{1}{2}$ can we say that $W + f$ is nowhere Hölder continuous with exponent α?

18 (*Hölder-$1/2$ points?)***.** We say that t_0 is a Hölder-$1/2$ point of f with constant C if $\limsup_{h\to 0} \frac{|f(t_0+h)-f(t_0)|}{\sqrt{h}} < C$. In this exercise, you will prove that almost surely, W has no Hölder-$1/2$ points with constant less than 0.1. Let $\Delta W(I) := W(b) - W(a)$ if $I = [a,b]$.

[**MP**] will indicate the book *Brownian motion* by Peter Mörters and Yuval Peres.

1. Fix $\delta > 0$ and set $\mathcal{A}_\delta$ be the event that there exists $t \in I$ such that $|W(t+h) - W(t)| \leq 0.1\ h^\delta$ for all $h \in [-\delta, \delta]$. The claim follows if we show that $\mathbf{P}(\mathcal{A}_\delta) = 0$ for any $\delta > 0$.
2. Let $I_{n,k} = [k2^{-n}, (k+1)2^{-n}]$. The parent of $I_{n,k}$ is the unique I_{n-1}, j that contains $I_{n,k}$. Now, fix m such that $2^{-m} < \delta$ and define $\mathcal{S}_m = \{I_{m,k} : 0 \leq k \leq 2^m - 1\}$. For $p > m$, define

$$\mathcal{S}_p = \{I_{p,k} : \text{ the parent of } I_{n,k} \text{ is in } \mathcal{S}_{p-1} \text{ and } |\Delta W(I_{n,k})| \leq 0.2\sqrt{2^{-n}}\}.$$

   If the “branching process” $\mathcal{S}_m, \mathcal{S}_{m+1}, \mathcal{S}_{m+2} \ldots$ becomes extinct almost surely, then $\mathbf{P}(\mathcal{A}_\delta) = 0$.
3. Use Problem 15 to calculate the “offspring probabilities” in the branching process and hence conclude that extinction happens almost surely.

Dvoretsky proved that almost surely W does have Hölder-$1/2$ points with constant C if C is large enough but not if C is small enough. The above proof shows the latter for $C < 0.1$ but a second look will show that we can improve this a little. The proof here is a modification of the original idea of Paley, Wiener and Zygmund where they proved nowhere differentiability. Observe that the proof of Dvoretsky, Erdös and Kakutani does not say anything about Hölder-$1/2$ points.