6. Markov and strong Markov property. (Reading material: *Sections 2.1-2.2 of* [**MP**]). Markov property may be thought of as a symmetry of a measure on $C_d := C([0,\infty);\mathbb{R}^d)$. For example, Markov property of Wiener measure is the symmetry of this measure on C_d under the operations $\theta_T : C_d \to C_d$ given by $\theta_T f(s) := f(T+s) - f(T)$, for any $T > 0$. Strong Markov property is the far more general symmetry that preserves Wiener measure under the transformations $\Theta_\tau f(s) := f(\tau_f + s) - f(\tau_f)$ where $\tau : C_d \to [0,\infty)$ is a function (any function will not do, τ has to be a stopping time). These symmetries are far more powerful than the simple symmetry under scaling, orthogonal transformations and time-reversal. Chapter 2 (and the rest of the book) has ample evidence for this statement.

28. Exercises 2.1, 2.2, 2.3 in [**MP**].

29.
1. Let W_0 be standard Brownian bridge. Let $X_t = (W_0(t), W_0(1-t))$ for $0 \le t < \frac{1}{2}$. Show that X is a Markov process and find its transition density $p_{t,s}((x_1,y_1),(x_2,y_2))$.
2. Let $\mathbf{W}$ be standard BM in d-dimensions and let $\mathbf{X}(t) = e^{-t/2}\mathbf{W}(e^t)$. Show that X is a stationary Markov process with transition kernel $q_{s,t}(\mathbf{x},\mathbf{y})$ for $s < t$ and $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$ given by
$$q_{s,t}(\mathbf{x},\mathbf{y})d\mathbf{y} = N_d(e^{-(t-s)/2}\mathbf{x}, (1-e^{(t-s)})I_d).$$

30. Show that there does not exist a stopping time τ for a 1-dim BM W such that τ is a local maximum of W with positive probability. (similar questions - Is there a stopping time such that τ is almost surely (a) a zero of W? (b) a Hölder-$1/2$ point of W with some small constant? etc.)

31. If $f : \mathbb{R}_+ \to (0,\infty)$ is any given function, show that $\limsup\limits_{t\to\infty} \frac{W_t}{f(t)} = c_f$, a constant in $[0,\infty]$, a.s. For $f(t) = t^\alpha$, show that $c_f = 0$ if $\alpha > \frac{1}{2}$ while $c_f = \infty$ for $\alpha \le \frac{1}{2}$.

32. Let $\mathbf{W}$ be d-dimensional BM with $d \ge 3$. For a Borel set $A \subseteq \mathbb{R}^d$, let $N_T(A) = \int_0^T \mathbf{1}_{\mathbf{W}_t \in A} dt$ be the *occupation measure* of the set A by the Brownian motion ($N_T(\cdot)$ is a random Borel measure). Let $p_t(x) = (2\pi)^{-d/2}\exp\{-\|x\|^2/2t\}$ be the transition density of Brownian motion.

1. Express $\mathbf{E}[N_T(A)]$ in terms of $p_t(x)$ and show that $(\frac{d}{2}-1)T^{\frac{d}{2}-1}\mathbf{E}[N_T(A)] \to \mathcal{L}(A)$ (the Lebesgue measure of the set A).
2. Show that $(\frac{d}{2}-1)^2 T^{d-2}\mathbf{E}[N_T(A)^2] \to \mathcal{L}(A)^2$.
3. Conclude that if $\mathcal{L}(A) > 0$, then with probability one, $\mathbf{W}$ hits A.
4. Modify the expressions for $d = 2$, but get the same conclusion that if $\mathcal{L}(A) > 0$, then with probability one, $\mathbf{W}$ hits A.