Example. $E = \frac{1}{3}$ Cantor set.

$\therefore\ |X-Y| \le \frac{2}{3^L} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{3^{L-1}}$

Y'lly $\quad 3^{-L} \le |X-Y|$

$\Rightarrow\ E[3^{(L-1)\alpha}] \le E[|X-Y|^{-\alpha}] \le E[3^{\alpha L}]$

$\quad \parallel$

$3^{-\alpha} E[3^{\alpha L}]$

For which α is this finite?

8-10-09

Now $\quad E[3^{\alpha L}] = \sum_{l=1}^{\infty} 3^{\alpha l}\, P[L=l]$

$$= \sum_{l=1}^{\infty} \left(\frac{3^\alpha}{2}\right)^l$$

Thus if $\alpha < \frac{\log 2}{\log 3}$ then $\frac{3^\alpha}{2} < 1$ & $\mathcal{E}_\alpha(\mu) < \infty$

$\Rightarrow\ \dim_{\mathcal{E}}(E) \ge \frac{\log 2}{\log 3}$

Thus $\frac{\log 2}{\log 3} \le \dim_{\mathcal{E}} \le \dim_H \le \underline{\dim}_M \le \frac{\log 2}{\log 3}$

$\Rightarrow\ \dim_H(E) = \underline{\dim}_M(E) = \frac{\log 2}{\log 3}$

Remark:— In general, to find $\dim_H(E)$ we find (a) Upper bound for $\overline{\dim}_M(E)$ (can be done by choosing any δ-cover)

(b) Lower bound for $\dim_{\mathcal{E}}(E)$

(any μ and $\alpha \geq 0$ s.t. $\mathcal{E}_\alpha(\mu) < \infty$, $\mu(E) = 1$ gives a lower bound of α for $\dim_{\mathcal{E}}(E)$)

Exercise*: Let E : Set of all $x \in [0,1]$ whose decimal expansion has digits only $1, 3, 5, 7, 9$.

Ex*: Find $\dim_H$ of the Sierpinski gasket

Example: For $x \in [0,1]$, write $x = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots$

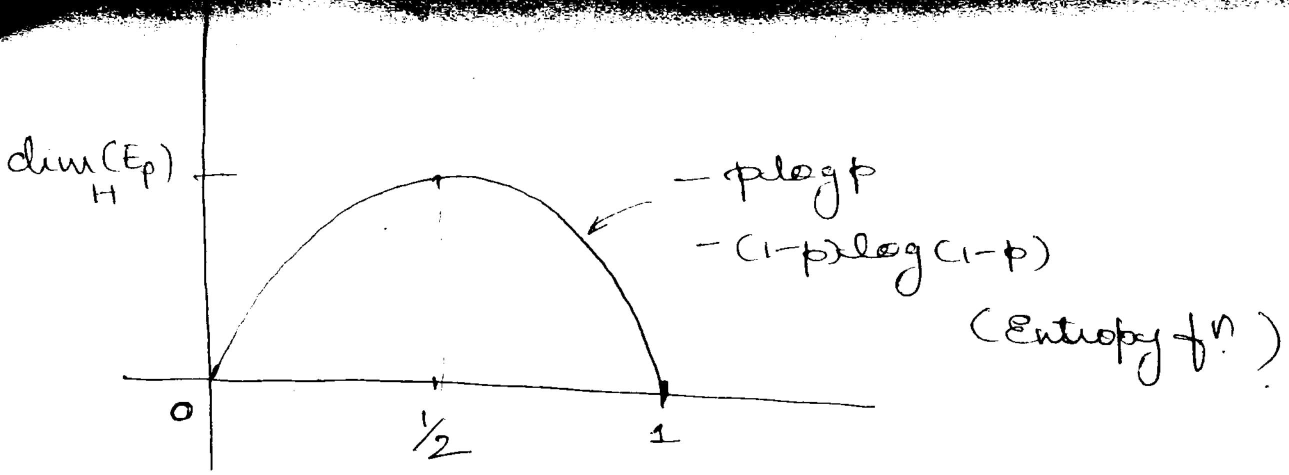
$x_i \in \{0,1\}$

Fix $p \subset [0,1]$

$$E_p = \{ x \in [0,1] \mid \lim_{n \to \infty} \frac{1}{2} \sum_{j=1}^{n} x_j = p \}$$

$$\mathrm{Leb}(E_p) = \begin{cases} 1 & \text{if } p = \frac{1}{2} \\ 0 & \text{if } p \neq \frac{1}{2} \end{cases}$$

But Lebesgue measure doesnot distinguish between E_p for $p \neq \frac{1}{2}$ $\dim_H$ does!



<u>Upper bound</u>: — Fix p. Let $\tilde{E}_p = \{x \mid \limsup \frac{1}{n}\sum_{j=1}^{n} x_j < p\}$

$$= \bigcup_{N=1}^{\infty} \tilde{E}_p(N)$$

where $\tilde{E}_p(N) = \{N \mid \frac{1}{n}\sum_{j=1}^{n} x_j < p, \forall n \geq N\}$

Now fix $p < \frac{1}{2}$, N and consider $\tilde{E}_p(N)$.

Let $\delta > 0$, Let $2^{-n} \leq \delta \leq 2^{-n+1}$

(If δ small, $n > N$).

Then let

$$S = \{\underbrace{(z_1, \cdots z_n)}_{\underline{z}} \in \{0,1\}^n \mid \sum_{i=1}^{n} z_i < np\}$$

Set $A_{\underline{z}} = \{\cancel{x} \in [0,1] \mid x_j = z_j, j \leq n\}$

for each $\underline{z} \in S_n$

Then $\bigcup_{\underline{\xi} \in S_n} A_{\underline{\xi}} \supseteq \tilde{E}_p(N)$ provided $n > N$

(true if δ small)

i.e. $\{A_{\underline{\xi}} \mid \underline{\xi} \in S_n\}$

is a δ-cover of $\tilde{E}_p(N)$

$$\Rightarrow \quad N_\delta(\tilde{E}_p(N)) \leq \# S_n = \sum_{k=0}^{np} \binom{n}{k}$$

$$\leq \binom{n}{np} np \overset{\text{Stirling's}}{\approx} \frac{n^{n+1/2} e^{-n} \sqrt{2\pi}\,(np)}{(np)^{np+1/2} e^{-np} \sqrt{2\pi}\,(nq)^{nq+1/2} e^{-nq} \sqrt{2\pi}}$$

$$= \frac{\sqrt{p}\; p^{-np} q^{-nq}}{\sqrt{2\pi nq}}$$

$$= \sqrt{\frac{p}{2\pi nq}}\; e^{-n[p\log p + q\log q]}$$

Hence $$\frac{\log N_\delta}{\log(1/\delta)} \lesssim \frac{-n[p\log p + q\log q] + O(\log n)}{n\log 2}$$

$$\longrightarrow -p\log_2 p - q\log_2 q$$

$$=: H(p)$$

Thus $\dim_H(\tilde{E}_p(N)) \leq \underline{\dim}_M(\tilde{E}_p(N)) \leq H(p)$

$$\tilde{E}_p = \bigcup_N \tilde{E}_p(N)$$

$$\Rightarrow \dim_H(\tilde{E}_p) = \sup_N \dim_H(\tilde{E}_p(N)) \le H(p)$$

(Using ctbl stability property of Haus-dim)

$E_p \subseteq \tilde{E}_{p'}$ $\forall p' > p$ $\Rightarrow \dim_H(E_p) \le H(p')$ $\forall p' > p$

$\Rightarrow \dim_H(E_p) \le H(p)$ ($\because$ $p \mapsto H(p)$ is cts)

<u>Lower bound</u>: $E_p = \{ x \in [0,1] \mid \frac{1}{n} \sum_{j=1}^{n} x_j \to p \text{ as } n \to \infty \}$

Let $X = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots$ where $x_j \sim$ iid $\mathrm{Ber}(p)$.

If $\mu = \mathcal{L}(X)$ then $\mu(E_p) = 1$

Then $\mathcal{E}_\alpha(\mu) = E[|X-Y|^{-\alpha}]$ $X, Y \sim$ iid μ

<u>Exer</u>*. Let $L = \min\{k \mid x_k \ne y_k\}$. Then

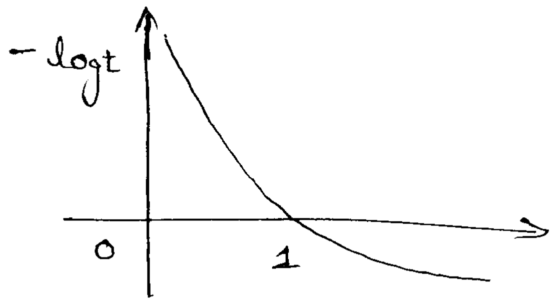
$|X - Y| \le 2^{-L+1}$

$\Rightarrow |X - Y|^{-\alpha} \geq 2^{(L-1)\alpha}$

~~Find all α for which~~

check that $E[2^{(L-1)\alpha}] = \infty$ for $\alpha > -\log_2(p^2+q^2)$

Hence $\mathcal{E}_\alpha(\mu) = \infty$ for $\alpha > -\log_2(p^2+q^2)$



Hence

$$-\log_2(p \cdot p + q \cdot q) \underset{p=q}{\overset{\text{strict unless}}{\le}} -p\log_2 p - q \log_2 q = H(p)$$

$\overline{\quad T} \Rightarrow 1, \alpha - X|$ mylo

Fix $M \geq 1$, a large integer,

Then let

$$S_M = \left\{ (\xi_1, \cdots, \xi_M) \,\middle|\, \sum_{i=1}^{M} \xi_i = \lfloor Mp \rfloor \right\}$$

Then take $Y_1, Y_2, \cdots$ iid uniform on S_M

i.e $P\{Y_1 = (\xi_1, \cdots, \xi_M)\} = \frac{1}{\# S_M}$ for each $(\xi_1, \cdots \xi_M) \in S_M$

Then let X = concatenation of $Y_1, Y_2, \cdots$

i.e if $Y_j = (\xi_{j1}, \cdots \xi_{jM})$ then

$$X = \left(\frac{\xi_{11}}{2} + \frac{\xi_{12}}{2^2} + \cdots \frac{\xi_{1M}}{2^M}\right) + \left(\frac{\xi_{21}}{2^{M+1}} \cdots\right) \cdots$$

Then $X \in E_p$ w.p.1 (check!)

Let $\mu = \mathcal{L}(X)$ so $\mu(E_p) = 1$

$$\mathcal{E}_\alpha(\mu) = E[|X - X'|^{-\alpha}]$$

$X, X' \sim$ iid μ.

X is made of $Y_1, Y_2, \cdots$ iid unif(S_M)

X' is made of $Y_1', Y_2' \cdots$ — " —

Let $L = \min\{k \mid Y_k \neq Y_k'\}$

for $M \geq 1$, a large integer

then $|\hat{x} - x'| \leq \frac{1}{2^{(L-1)M}}$

Consider S_M, then let

$$Y_* = \big(\underbrace{\tfrac{0}{2} + \cdots + \tfrac{0}{2^{Mq}}}_{Mq} + \underbrace{\tfrac{1}{2^{Mq+1}} + \cdots + \tfrac{1}{2^{M}}}_{Mp}\big) \in S_M$$

$$Y^* = \big(\underbrace{\tfrac{1}{2} + \cdots + \tfrac{1}{2^{Mp}}}_{Mp} + \underbrace{\tfrac{0}{2^{Mp+1}} + \cdots + \tfrac{0}{2^{M}}}_{Mq}\big) \in S_M$$

Then $\forall\, Y \in S_M$

$$Y_* \leq Y \leq Y^*$$

$$\therefore\ |X - X'| \geq \frac{1}{2^{LM}} - \left(\frac{Y^* - Y_*}{2^{LM}} + \frac{Y^* - Y_*}{2^{(L+1)M}} + \cdots\right)$$

$$\geq \frac{c}{2^{LM}} \qquad c > 0.$$

$$E\left[|X - X'|^{-\alpha}\right] \leq E\left[2^{LM\alpha}\right] c^{-\alpha}$$

$L \sim$ Geometric

$$P[L = l] = \frac{1}{(\#S)^{l-1}}\left(1 - \frac{1}{\#S}\right)$$

Find α for which $E\left[2^{LM\alpha}\right] < \infty$

Then let $M \to \infty$.

Sec: 19 Brownian fractals :—

13-10-2009

1) 1 dim B.M: Let $G_B = \{(t, B_t) \mid 0 \le t \le 1\}$

In this case $\dim_H(G_B) = \frac{3}{2}$ a.s.

2) d dim B.M: Let $Im_B = \{B_t \mid t \in [0,1]\}$

$$\dim_H(Im_B) = \begin{cases} 1, & d = 1 \\ 2, & d \ge 2 \end{cases}$$

3) 1 dim B.M: $Z_B = \{t \mid B_t = 0,\ t \in [0,1]\}$

|| Z_B is a perfect set! ||

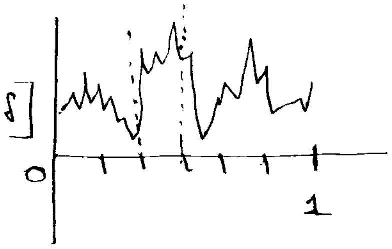
$$\dim_H(Z_B) = \frac{1}{2} \text{ a.s.}$$

Proof of 1) : Graph of 1-dim B.M.

Upper bound: Let $f : [0,1] \to \mathbb{R}$ be a cts function.

Suppose f is Hölder γ

Fix $\delta > 0$. Divide $[0,1] = \bigcup_{k=0}^{1/\delta} [k\delta, (k+1)\delta]$



Now f is Hölder γ

$\Rightarrow |f(x) - f(y)| \le c|x-y|^\gamma$

Fix $k \le \frac{1}{\delta}$

Then $|f(x) - f(y)| \le c\delta^\gamma,\ \forall x, y \in [k, (k+1)\delta]$

Sec:19 Brownian paths: — 13-10-2009

Thus $\{(t, f(t)) \mid t \in [k\delta, (k+1)\delta]\}$ can be covered by $\frac{c\delta^{\gamma}}{\delta}$ squares of sidelength δ each.

Hence $N_{\delta\sqrt{2}}(G_f) \leq \frac{c\delta^{\gamma}}{\delta}\frac{1}{\delta} = c\delta^{\gamma-2}$

$\Rightarrow \overline{\dim}_M(G_f) \leq 2-\gamma.$

B is Holder γ $\forall$ $\gamma < \frac{1}{2}$ (a.s.)

$\Rightarrow \dim_M(G_B) \leq 2-\frac{1}{2} = \frac{3}{2}$ a.s.

Lower bound: Let $\phi(t) = (t, B(t))$

$\phi : [0,1] \to G_B$

(1-1, onto).

Let $\mu = \text{Leb}_{[0,1]} \circ \phi^{-1}$: a prob measure on G_B

consider

$$\mathcal{E}_\alpha(\mu) = \int_{G_B}\int_{G_B} \frac{d\mu(x)\,d\mu(y)}{\|x-y\|^\alpha_{\mathbb{R}^2}}$$

$$= \int_0^1\int_0^1 \frac{dt\,ds}{\|(t,B_t)-(s,B_s)\|^\alpha_{\mathbb{R}^2}}$$

$$= \int_0^1\int_0^1 \frac{dt\,ds}{\left[(t-s)^2+(B_t-B_s)^2\right]^{\alpha/2}}$$

$$E\left[e^{-\theta X^2}\right] = \int e^{-\theta x^2} e^{-x^2/2}\, dx$$

$$E\left[\mathcal{E}_\alpha(\mu)\right] = \int_0^1\int_0^1 E\left[\frac{1}{[(t-s)^2+(B_t-B_s)^2]^{\alpha/2}}\right] dt\, ds$$

For fixed t, s.

$$E\left[\frac{1}{[(t-s)^2+(B_t-B_s)^2]^{\alpha/2}}\right]$$

$$= \frac{1}{(t-s)^\alpha}\, E\left[\frac{1}{(1+a X^2)^{\alpha/2}}\right]$$

where $a = \dfrac{1}{|t-s|}$ & $X \sim N(0,1) \longrightarrow \ddagger$

Now we know that

$$\int_0^\infty \frac{u^{\beta-1}\cdot e^{-\lambda u}\, du}{\Gamma(\beta)} = \frac{1}{\lambda^\beta}$$

Hence

$$E\left[\frac{1}{(1+a X^2)^\beta}\right] = E\left[\int_0^\infty \frac{u^{\beta-1} e^{-u(1+aX^2)}}{\Gamma(\beta)}\, du\right]$$

$$= \frac{1}{\Gamma(\beta)}\int_0^\infty u^{\beta-1} e^{-u}\, E\left[e^{-ua X^2}\right] du$$

$*$

$$E\left[e^{-\theta X^2}\right] = \int_{\mathbb{R}} e^{-\theta x^2} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}(1+2\theta)}\, dx$$

$$= \frac{1}{\sqrt{1+2\theta}}$$

$$\leq \frac{1}{\sqrt{2\theta}}.$$

$\therefore$ From $*$, $E\left[\dfrac{1}{(1+aX^2)^\beta}\right]$

$$\leq \frac{1}{\sqrt{a}\,\Gamma(\beta)\sqrt{2}} \int_0^\infty u^{\frac{\alpha-1}{2}-1} e^{-u}\, du$$

$$= \frac{C_\alpha}{\sqrt{a}} \quad \text{where } C_\alpha = \frac{\Gamma\left(\frac{\alpha-1}{2}\right)}{\sqrt{2}\,\Gamma(\alpha/2)} \qquad \left\| \text{where } \beta = \frac{\alpha}{2} \right\|$$

From $\ddagger$, $E\left[\dfrac{1}{\left[(t-s)^2 + (B_t - B_s)^2\right]^{\alpha/2}}\right] \leq \dfrac{C_\alpha}{|t-s|^{\alpha-\frac{1}{2}}}$

$\Rightarrow E\left[\mathcal{E}_\alpha(\mu)\right] \leq C_\alpha \displaystyle\int_0^1\int_0^1 \frac{dt\, ds}{|t-s|^{\alpha-1/2}} < \infty$ if $\alpha < \frac{3}{2}$

$\Rightarrow \mathcal{E}_\alpha(\mu) < \infty$ a.s if $\alpha < \frac{3}{2}$

$\Rightarrow \dim_{\mathcal{E}}(G_B) \geq \frac{3}{2}$. Hence $\dim_H(G_B) = \frac{3}{2}$.

2> Image of d-dim B.M. $(d \geq 2)$

Upper bound: Fix $\delta > 0$

Then consider pts $t_k = k\delta^{\frac{1}{\gamma}}$ where $\gamma < \frac{1}{2}$

and $k = 0, 1, 2, \ldots \frac{1}{\delta^{1/\gamma}}$

B is Hölder-γ $\Rightarrow$ $|B(t) - B(t_k)| \leq C(\delta^{1/\gamma})^{\gamma}$

$= C\delta$

$t \in [t_{k-1}, t_{k+1}]$

where C = Hölder γ-constant, (random but finite a.s.).

Therefore balls of radius $C\delta$ around $B(t_k)$, $1 \leq k \leq \frac{1}{\delta^{1/\gamma}}$ give a covering of $Im(B)$.

$\Rightarrow N_{C\delta}(Im_B) \leq \frac{1}{\delta^{1/\gamma}}$

$\Rightarrow \underline{\dim}_M(Im_B) \leq \frac{1}{\gamma}$ true for all $\gamma < \frac{1}{2}$

$\Rightarrow \underline{\dim}_M(Im_B) \leq 2$ a.s.

Lower bound: — $B : [0,1] \longrightarrow Im(B)$

Let $\mu = Leb_{[0,1]} \circ B^{-1}$ $\rightarrow$ occupation measure of B.M.

i.e. $\mu(A)$ = length of time spent in A by B.M. $= \int_0^1 1_A(B_s)\,ds$.

2> Image of d-dim B.M. (d≥2)

$$\mathcal{E}_\alpha(\mu) = \int_0^1\int_0^1 \frac{dt\,ds}{\|B_t - B_s\|_{\mathbb{R}^d}}$$

$$E(\mathcal{E}_\alpha(\mu)) = \int_0^1\int_0^1 E\left[\frac{1}{\|B_t - B_s\|^\alpha_{\mathbb{R}^d}}\right] dt\,ds$$

For fixed t, s, $B_t - B_s \sim N_d(0, |t-s| I_d)$

$$\stackrel{d}{=} \sqrt{|t-s|}\,V$$

where $V \sim N_d(0, I_d)$.

$$\Rightarrow E\left[\frac{1}{\|B_t - B_s\|^\alpha}\right] = \frac{1}{|t-s|^{\alpha/2}} E\left[\frac{1}{\|V\|^\alpha}\right]$$

Thus

$$E[\mathcal{E}_\alpha(\mu)] = \underbrace{E\left[\frac{1}{\|V\|^\alpha}\right]}_{\substack{<\infty \\ \forall \alpha < d}} \underbrace{\int_0^1\int_0^1 \frac{dt\,ds}{|t-s|^{\alpha/2}}}_{\substack{<\infty \\ \forall \alpha < 2}}$$

$$\left\| \because \int_{B_d(0,1)} \frac{1}{\|x\|^\alpha} < \infty \quad \forall \alpha < d. \right\| \leftarrow$$

Thus in $d \geq 2$ $\forall \alpha < 2$, $\mathcal{E}_\alpha(\mu) < \infty$ a.s.

$\Rightarrow \dim_{\mathcal{E}}(\mathrm{Im}_B) \geq 2$ a.s

$\Rightarrow \dim_H(\mathrm{Im}(B)) = 2$

5.) Zero set of 1-dim B.M.

<u>U.B</u> Exer* $\longrightarrow$ Take $Z_B = \{ t \in [1,2] \mid B_t = 0 \}$

(Hint: & then take expectation)

<u>L.B</u> Levy's identity say $M - B \stackrel{d}{=} |B|$

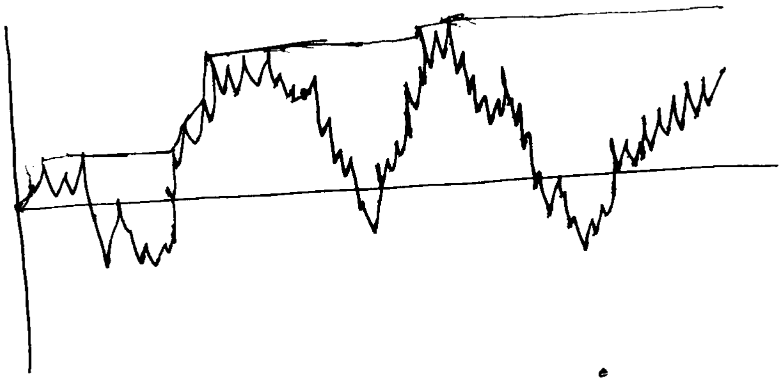
$$\left.\begin{array}{l} Z_B = \{ t \mid |B_t| = 0 \} \\ Rec_B = \{ t \mid M_t - B_t = 0 \} \end{array}\right\rangle \quad Z_B \stackrel{d}{=} Rec_B$$

$\downarrow$

Record times.

In particular

$\dim_H(Z_B) = \dim_H(Rec_B)$



Consider M_t : 1) Increasing

2) M_t is constant on any interval that does not intersect Rec_B.

clearly M is the distribution function of a measure that is supported on Rec_B.

5) Zero set of 1-dim B.M.

15-10-2009

Define μ = measure whose distribution fn.

$$\mu[a,b] = M(b) - M(a).$$

Then (1) μ is supported on Rec_B

(2) $\mu[0,1] > 0$ (a.s.)
$= M(1)$

(3) Fix $\alpha < \frac{1}{2}$. Then $\exists\, C < \infty$ a.s s.t
C (random)

$$|B_t - B_s| \le C|t-s|^\alpha \quad \forall\, s,t \in [0,1]$$

Then $M_t - M_s \le C(t-s)^\alpha \quad \forall\, s < t \le 1$

because $B_s \le M_s$ & $B_{t'} = M_t$ for some $t' \le t$
where $t' = \sup\{u \le t \mid B_u = M_t\}$

Hence $$M_t - M_s \begin{cases} = 0 & \text{if } t' < s \\ \le |B_{t'} - B_s| \le C(t'-s)^\alpha & \text{if } t' \in [s,t] \\ \quad \le C(t-s). & \end{cases}$$

Now consider the α-Hausdorff measure of Rec_B

$$H_\alpha(Rec_B) = \lim_{\delta \downarrow 0} \inf_{\substack{\{A_j\} \\ \delta \text{ cover of } Rec_B}} \sum_j |A_j|^\alpha$$

Firstly note that it is enough to take infimum over δ-cover $\{A_j\}$ that are

(i) $A_j = (a_j, b_j)$ intervals

(ii) $A_j \cap A_k = \phi$ if $j \neq k$.

For such a δ cover, consider

$$\sum_j |A_j|^\alpha = \sum_j (b_j - a_j)^\alpha$$

(if $\alpha < \frac{1}{2}$ by property 3) $\Rightarrow \geq \frac{1}{C_\alpha} \sum_j [M(b_j) - M(a_j)]$

$$= \frac{1}{C_\alpha} \sum_j \mu(A_j)$$

$$= \frac{1}{C_\alpha} \mu(\mathrm{Rec}_B)$$

$$= \frac{M(1)}{C_\alpha}$$

Hence $H_\alpha(\mathrm{Rec}_B) \geq \frac{M(1)}{C_\alpha} > 0$ (a.s) for each $\alpha < \frac{1}{2}$

$\Rightarrow \dim_H(\mathrm{Rec}_B) \geq \frac{1}{2}$.

<u>Fact about Z_B</u> : Almost surely, Z_B is perfect, i.e. every pt of Z_B is an accumulation pt of Z_B

<u>Proof</u> :— Fix $q \geq 0$ and let $Z_q = \inf\{t \geq q \mid B_t = 0\}$

Clearly Z_q is a stopping time.

By SMP $(B(Z_q + t) - B(Z_q))_{t \geq 0}$ is std B.M

$= (B(Z_q + t))_{t \geq 0}$

Therefore w.p. 1 $\exists\, t_n \downarrow 0$ s.t $B(Z_q + t_n) = 0 \ \forall n$

$t_1 > t_2 > \cdots t_n > \cdots 0$.

$\Rightarrow Z_q$ is an accumulat^n pt of Z_B a.s.

Z_q is a accum[illegible] $Q_+ \} = 1$

<u>Note:</u> Each Z_q, $q \in \mathbb{Q}_+$ accumulation pt from the right.

Now consider any $t_* \in Z_B$

Then $t_* = Z_q$ for some $q \in \mathbb{Q}_+ \cap [0,t]$ or

t_* is an accumulation pt of Z_B from the left.

$\boxtimes$

|| Frontier / outer boundary of B. M. has $\overset{H.D}{4/3}$ ||

|| Q: What are the extra terms to B_t^3 to make it a martingale? ||