Arcsine law for time spent by a SRW on $[0,\infty)$

Let $X_1, X_2, \ldots$ be iid $+1$ or -1 w.p. $\frac{1}{2}$ each.

$S_k = X_1 + \ldots + X_k$.

$$p_{2k,2n} = \#\{ j \mid 0 \le j \le 2n-1,\ [S_j, S_{j+1}] \subseteq [0,\infty) \}$$

$$= \#\text{ of steps taken on } [0,\infty)$$

NOTE: THIS IS THE ONLY POINT OF DIFFERENCE FROM WHAT WE DID IN CLASS

$$f_{2k} = P\{ S_{2k} = 0,\ S_j \neq 0 \text{ for } j \le 2k-1 \}$$

$$u_{2k} = P\{ S_{2k} = 0 \} = \binom{2k}{k} \frac{1}{2^{2k}}$$

WITH THIS DEFINITION, THE PROBLEM OF ODD AND EVEN DO NOT ARISE. i.e., # steps taken on $[0,\infty)$ upto time $2n$ is necessarily even.

<u>Lemma:</u> $p_{2k,2n} = u_{2k}\, u_{2n-2k}$,

<u>Proof:</u>

$$p_{2k,2n} = \frac{1}{2} \sum_{r=1}^{k} f_{2r}\, p_{2k-2r,\,2n-2r} + \frac{1}{2} \sum_{r=1}^{n-k} f_{2r}\, p_{2k,\,2n-2r}$$

prob. that the first step $X_1 > 0$

prob. to return to zero at $2r$ for the first time

prob. that $X_1 < 0$

same as on the left.

In the first $2r$ steps, all of them were spent in $[0,\infty)$ so we need $2k-2r$ more steps in $[0,\infty)$ out of a total of $2n-2r$ steps.

In the first $2r$ steps, none were in $[0,\infty)$, so we need to take $2k$ steps in $[0,\infty)$ in the remaining $2n-2r$ steps.

Then inductively assuming $p_{2j,2m} = u_{2j}\, u_{2m-2j}$ $\forall\, m \le n-1$ $\forall\, j \le m$, and using $\sum_{r=1}^{m} f_{2r}\, u_{2m-2r} = u_{2m}$, we get

$$p_{2k,2n} = \frac{1}{2} u_{2n-2k} \sum_{r=1}^{k} f_{2r}\, u_{2k-2r} + \frac{1}{2} u_{2k} \sum_{r=1}^{n-k} f_{2r}\, u_{2n-2k-2r}$$

$$= \frac{1}{2} u_{2n-2k}\, u_{2k} + \frac{1}{2} u_{2k}\, u_{2n-2k}$$

$$= u_{2k}\, u_{2n-2k}. \quad \blacksquare$$

The rest is unchanged, except that we get $\dfrac{\#\text{ steps in } [0,\infty) \text{ by } S_0, S_1, \ldots, S_n}{n} \xrightarrow{d} \text{arcsine}$

and then one can also get $\dfrac{\#\{ j \le n \mid S_j \ge 0 \}}{n} \xrightarrow{d} \text{arcsine}$.