X_n - a transient Markov chain on a countable state space S.

$$G(x,y) = E_x\left[\sum_{n\ge 0}^{\infty} 1(X_n=x)\right] = \sum_{n\ge 0}^{\infty} p^{(n)}(x,y) \qquad \text{[Green kernel]}$$

for $x,y\in S$.

$$M(x,y) = \frac{G(x,y)}{G(0,y)} \qquad 0\in S \text{ a fixed reference point.}$$

[Martin Kernel]

Fix $A\subseteq S$. Then

$$\tfrac{1}{2}\mathrm{Cap}_M(A) \le P_0\{X \text{ hits } A\} \le \mathrm{Cap}_M(A)$$

Lower bound: Enough to show that for any p.m μ supported on A

$$P_0\{X \text{ hits } A\} \ge \frac{1}{2\mathcal{E}_M(\mu)}$$

where $\mathcal{E}_M(\mu) = \sum M(x,y)\mu(x)\mu(y)$.

Define $$Z = \sum_{y\in A}\sum_{n\ge 0}^{\infty}\frac{1(X_n=y)}{G(0,y)}\mu(y) \rightarrow$$

NOTE THAT THIS Z IS DIFFERENT FROM WHAT WE CHOSE IN CLASS.

Then $$E_0[Z] = \sum_{y\in A}\frac{\mu(y)}{G(0,y)}\cdot\sum_{n\ge 0}^{\infty}P_0(X_n=y) = \sum_{y\in A}\mu(y) = 1$$

because $\sum_{n\ge 0}^{\infty} P_0(X_n=y) = G(0,y)$ by definition.

In defining this we assume that X is irreducible, so that $G(0,y)>0$.

Further, $$E_0[Z^2] = \sum_{y,z\in A}\underbrace{\sum_{n,m\ge 0}\frac{P_0(X_n=y,\ X_m=z)}{G(0,y)\,G(0,z)}}_{I(y,z)}\mu(y)\mu(z)$$

Consider the inner sum $I(y,z)$

$$I(y,z) \le 2\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{P_0(X_n=y,\ X_m=z)}{G(0,y)\,G(0,z)}$$

$$= 2\,\frac{\sum_{n\ge 0}^{\infty}P_0(X_n=y)\sum_{m\ge n}^{\infty}P_0(X_m=z\mid X_n=y)}{G(0,y)\,G(0,z)}$$

$$= 2\,\frac{G(y,z)\,G(0,y)}{G(0,y)\,G(0,z)} \qquad \left(\because \sum_{m\ge n}^{\infty}P_0(X_m=z\mid X_n=y) = \sum_{k\ge 0}^{\infty}P_y(X_k=z) = G(y,z)\right)$$

$$= 2\,M(y,z)$$

Hence $$E_0[Z^2] \le 2\sum_{y,z\in A}M(y,z)\mu(y)\mu(z) = 2\,\mathcal{E}_M(\mu).$$

Finally, by the second moment method

$$P_0\{X \text{ hits } A\} = P_0\{Z>0\} \ge \frac{(E[Z])^2}{E[Z^2]} = \frac{1}{2\mathcal{E}_M(\mu)}.$$

Upper bound: Exercise! (Use the hitting measure of A by X)