KARHUNEN-LOÈVE EXPANSION

Theorem Let χ_n be i.i.d. standard Normal variables. Then, almost surely, the series

$$W_t := \sum_{n=0}^{\infty} \chi_n \frac{\sqrt{2}\sin(n\pi t)}{n\pi}$$

converges uniformly for $t \in [0,1]$ and then W is the standard Brownian bridge.

For any fixed t, the sequence $\sin(n\pi t)/\pi n$ is in ℓ^2, and hence the series defining B_t converges a.s. We need the following lemma to prove uniform convergence. It is a weaker form of a famous inequality of Bernstein that asserts that $\|p'\|_{L^\infty} \le N\|p\|_{L^\infty}$.

Lemma Let $p(t) = \sum_{k=0}^{N-1} c_n \sin(kt)$ (more genrally, any trigonometric polynomial of degree at most N). Then (i) $\|p'\|_{L^\infty} \le N^2\|p\|_{L^\infty}$. (ii) There is an interval of length $1/N^2$ on which $|p(t)| \ge \frac{1}{2}\|p\|_{L^\infty}$.

Proof (i) Clearly

$$\|p'\|_{L^\infty} = \max_{0\le t\le 1}\left|\sum_{n=0}^{N-1} c_n n\pi \sin(n\pi t)\right| \le \left(\max_{0\le n\le N-1}|c_n|\right)\frac{\pi}{2}N(N-1).$$

By the orthogonality of $\sin(n\pi t)$ on $[0,1]$, and $\int \sin^2(n\pi t)dt = \frac{1}{2}$, we see that

$$|c_n| = \frac{1}{2}\left|\int p(t)\sin(n\pi t)dt\right| \le \frac{1}{2}\|p\|_{L^\infty}$$

from which we get $\|p'\|_{L^\infty} \le \frac{\pi}{4}N(N-1)\|p\|_{L^\infty} \le N^2\|p\|_{L^\infty}$.

(ii) Thus, if $|p(t_*)| = \|p\|_{L^\infty}$, then for all $|t-t_*| \le \frac{1}{2N^2}$, part (i) implies that $|p(t)-p(t_*)| \le \|p'\|_{L^\infty}|t-t_*| \le \frac{1}{2}\|p\|_{L^\infty}$. Thus $|p(t)| \ge \frac{1}{2}\|p\|_{L^\infty}$ on the interval $[t_*-1/2N^2, t_*+1/2N^2]$ which has length $1/N^2$. □

Proof[Theorem] Fix $k \ge 1$ and consider $p_k(t) := \sum_{n=2^k}^{2^{k+1}-1} \chi_n \frac{\sqrt{2}\sin(n\pi t)}{n\pi}$. We would like to get an upper bound for the sup norm of p_k. By the lemma, we are assured of an interval of length 2^{-2k-2} on which p_k is at least half of $\|p_k\|_{L^\infty}$. Therefore, for any $\lambda > 0$, we get

$$\int_0^1 (e^{\lambda p_k(t)} + e^{-\lambda p_k(t)})dt \ge \frac{1}{2^{2k+2}} e^{\frac{1}{2}\lambda\|p_k\|_{L^\infty}}.$$

Now take expectations over χ_ns to get

$$\mathbf{E}\left[e^{\frac{1}{2}\lambda\|p_k\|_{L^\infty}}\right] \le 2^{2k+2}\int_0^1 \mathbf{E}\left[e^{\lambda p_k(t)} + e^{-\lambda p_k(t)}\right]dt = 2^{2k+3}\int_0^1 \exp\left\{\lambda^2 r_k(t)\right\}dt$$

where $r_k(t) = \sum_{n=2^k}^{2^{k+1}-1} \frac{\sin^2(n\pi t)}{n^2\pi^2}$ by the well known $\mathbf{E}[e^{a\chi}] = e^{a^2/2}$. Clearly $r_k(t) \le \frac{1}{\pi^2 2^k}$. Therefore, we get $\mathbf{E}\left[e^{\frac{1}{2}\lambda\|p_k\|_{L^\infty}}\right] \le 2^{2k+3}\exp\left\{\frac{\lambda^2}{\pi^2 2^k}\right\}$. By Markov's inequality, it follows that $\mathbf{P}\left[\|p_k\|_{L^\infty} \ge x\right] \le 2^{2k+3}\exp\left\{\frac{\lambda^2}{\pi^2 2^k} - \lambda x\right\}$. With $x = 2^{-k/4}$ and $\lambda = 2^{k/2}$, we get

$$\mathbf{P}\left[\|p_k\|_{L^\infty} \ge 2^{-k/4}\right] \le 2^{2k+3}\exp\left\{\frac{1}{\pi^2} - 2^{k/4}\right\}$$

which is rapidly decaying in k and hence by Borel Cantelli, we see that almost surely, $\|p_k\|_{L^\infty} \le 2^{-k/4}$ for all large k. This implies that $W_t = \sum_k p_k(t)$ is uniformly convergent for $t \in [0,1]$, a.s.

From the uniform convergence it follows that W is a.s. a continuous function on $[0,1]$. It is also a Gaussian process since χ_n are i.i.d. Normals. To show that W is the Brownian bridge, it suffices to show that its covariance kernel

$$\sum_{n=1}^{\infty} \frac{2\sin(n\pi t)\sin(n\pi s)}{\pi^2 n^2} = \begin{cases} t(1-s) & \text{if } t < s. \\ s(1-t) & \text{if } s < t. \end{cases}$$

We showed this in class (try a direct proof!). □