Claim: LIL for BM $\Rightarrow$ LIL for RW ($EX_i=0$, $EX_i^2=1$)

a.s., $\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1$ a.s., $\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1$

Proof: $\tau_1 < \tau_2 < \dots$ stopping times so that $(S_n)_{n\geq 1} \overset{d}{=} (B(\tau_n))_{n\geq 1}$

Claim: $$\lim_{t\to\infty} \frac{|B(\tau_{\lfloor t\rfloor}) - B(t)|}{\sqrt{2t\log\log t}} = 0 \quad -(\star)$$

Work it out!

Assume the claim. Clearly $\limsup_{t\to\infty} \frac{B(\tau_{\lfloor t\rfloor})}{\sqrt{2\lfloor t\rfloor \log\log \lfloor t\rfloor}} = \limsup_{t\to\infty} \frac{B(\tau_{\lfloor t\rfloor})}{\sqrt{2t\log\log t}}$ ~~$\leq \limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t\log\log t}}$~~

~~$\exists B$~~

Pick ~~$\exists$~~ $t_k \to \infty$ s.t. $\frac{B(t_k)}{\sqrt{2t_k\log\log t_k}} \to 1$. Then $\frac{B(\tau_{\lfloor t_k\rfloor})}{\sqrt{2\lfloor t_k\rfloor \log\log\lfloor t_k\rfloor}} \to 1 \dots$

Remain to prove $(\star)$: $\frac{\tau_n}{n} = \frac{1}{n}\sum_{k=1}^{n} \tau_k - \tau_{k-1} \longrightarrow E[X_1^2] = 1$

Let $\delta > 0$. Fix $\epsilon > 0$ small enough so that $\boxed{\gamma = \frac{\delta^2}{4[(1+\epsilon)^3 - 1]} > 1}$ $-(\star\star)$

Then $\exists\, T_\epsilon(\omega) < \infty$ (a.s.) s.t. $\forall\, t \geq T_\epsilon(\omega)$, $\frac{1}{1+\epsilon} \leq \frac{\tau_{\lfloor t\rfloor}(\omega)}{t} \leq 1+\epsilon$

Fix $t_n = (1+\epsilon)^n$, $n = 1, 2, \dots$

Consider
$$\sup_{t_n \leq t \leq t_{n+1}} \frac{|B(\tau_{\lfloor t\rfloor}) - B(t)|}{\sqrt{2t\log\log t}} \leq \sup_{t_{n-1}\leq t, s \leq t_{n+2}} \frac{|B(s) - B(t)|}{\sqrt{2t\log\log t}} \quad \begin{pmatrix} \text{if } T_\epsilon < t_n \\ t_{n-1} < \tau_{\lfloor t\rfloor} < t_{n+2} \end{pmatrix}$$
$$\leq \frac{2 \cdot \sup_{t_{n-1}\leq t \leq t_{n+2}} |B(t) - B(t_{n-1})|}{\sqrt{2t_{n-1}\log\log t_{n-1}}}$$

This holds $\forall n$ s.t $t_n > T_\epsilon(\omega)$. Since $T_\epsilon < \infty$ (a.s.),

$$\limsup_{t\to\infty} \frac{|B(\tau_{\lfloor t\rfloor}) - B(t)|}{\sqrt{2t\log\log t}} \leq \limsup_{n\to\infty} \frac{2 \sup_{t_{n-1}\le t\le t_{n+2}} |B(t) - B(t_{n-1})|}{\sqrt{2t_{n-1}\log\log t_{n-1}}} \quad \text{—} \ (1)$$

We show that the RHS $\leq \delta$. For,

$$P\left\{ \sup_{t_{n-1}\le t\le t_{n+2}} |B(t) - B(t_{n-1})| > \frac{1}{2}\delta\sqrt{2t_{n-1}\log\log t_{n-1}} \right\}$$

$$\leq 4\, P\left\{ X\sqrt{t_{n+2} - t_{n-1}} > \frac{\delta}{2}\sqrt{2t_{n-1}\log\log t_{n-1}} \right\} \qquad \left(\because\ M(t) \stackrel{d}{=} |B(t)| \stackrel{d}{=} \sqrt{t}\,X. \quad X \sim N(0,1) \right)$$

$$\leq 4 \cdot \exp\left\{ -\frac{1}{2}\cdot\frac{\delta^2}{4}\cdot 2t_{n-1}\log\log t_{n-1} \cdot \frac{1}{t_{n-1}\left(\frac{t_{n+2}}{t_{n-1}} - 1\right)} \right\}$$

$$= 4\cdot\exp\left\{ -\frac{\delta^2}{4}\cdot\frac{\log(n-1) + \log\log(1+\epsilon)}{(1+\epsilon)^3 - 1} \right\}$$

$$= \frac{4}{[\log(1+\epsilon)]^{\frac{\delta^2}{4[(1+\epsilon)^3-1]}}} \cdot \frac{1}{n^{\frac{\delta^2}{4}\frac{1}{(1+\epsilon)^3-1}}}$$

$$= \frac{C_{\delta,\epsilon}}{n^\gamma} \qquad (\gamma > 1 \text{ - see } (**))$$

Hence by Borel-Cantelli, we get $\displaystyle\limsup_{n\to\infty} \frac{2\sup_{t_{n-1}\le t\le t_{n+2}} |B(t) - B(t_{n-1})|}{\sqrt{2t_{n-1}\log\log t_{n-1}}} \leq \delta$ a.s.

By (1), we get

$$\limsup_{t\to\infty} \frac{|B(\tau_{\lfloor t\rfloor}) - B(t)|}{\sqrt{2t\log\log t}} \leq \delta$$

LHS has no δ, hence $= 0$. This proves the claim.