The proof of Theorem 16.10 that you will find in either of the above references is specific to the Brownian situation, of course. But there is a sense in which this result can be seen as part of the much more general theory of *large deviations*, pioneered by Cramér, Schilder, Donsker and Varadhan, Ventcel and Freidlin, and developed further by Donsker, Varadhan, Stroock, among others. The excellent account by Deuschel and Stroock [1] contains a proof of Theorem 16.10 from a large-deviations point of view, and is delightfully clear.

3. BROWNIAN MOTION IN HIGHER DIMENSIONS

17. Some martingales for Brownian motion. By Brownian motion in $\mathbb{R}^d$ we mean a process $B_t := (B_t^1, \ldots, B_t^d)$ where each of the $(B_t^j)_{t\geq 0}$ $(j = 1, \ldots, d)$ is a Brownian motion, independent of all the others. To study Brownian motion in $\mathbb{R}^d$, we are going to need martingales, and the purpose of this section is to derive a result that gives us all the martingales we shall need. This result can be seen as a special case of general results in Markov process theory or in stochastic calculus, but we shall prove it here using the special structure of Brownian motion, since we do not yet have the general results.

(17.1) THEOREM. *Suppose that* $f: \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ *is* $C^{1,2}$, *and that there exists a constant K such that, for all* $t \geq 0$, $x \in \mathbb{R}^d$,

$$(17.2)\quad |f(t,x)| + \left|\frac{\partial f}{\partial t}(t,x)\right| + \sum_{j=1}^d \left|\frac{\partial f}{\partial x_j}(t,x)\right| + \sum_{i=1}^d \sum_{j=1}^d \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x)\right| \leq K e^{K(t+|x|)}$$

Then the process

$$(17.3)\quad C_t^f := f(t,B_t) - f(0,B_0) - \int_0^t \mathscr{G} f(s,B_s)\,ds \quad \text{is a martingale,}$$

where

$$(17.4)\quad \mathscr{G} f(t,x) := \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2}\right)(t,x).$$

Remarks. The class $C^{1,2}$ is, of course, the class of functions $f(t,x)$ with continuous partial derivatives of all orders up to 1 in t and up to 2 in x. The exponential growth condition (17.2) will be seen to be unnecessary provided we relax the statement (17.3) to say that C^f is a *local* martingale. We shall not digress to define this now. In dimension $d = 1$, the only functions of x for which $f(B_t)$ is a martingale are the linear functions, but in dimension $d \geq 2$ we shall see that there is a very rich family of f for which $f(B_t)$ is a martingale.

Proof. We must prove that, for $0 \leq s \leq t$,

$$\mathbf{E}[C_t^f - C_s^f \mid \mathscr{F}_s] = 0,$$

for which, by the independent-increments property of B, it will suffice to prove that, for any $x \in \mathbb{R}^d$ and $t \geq 0$

$$(17.5)\quad \mathbf{E}^x[C_t^f] = 0,$$

where $\mathbf{P}^x$ is the law of Brownian motion started at x. Without loss of generality, we can take $x = 0$ (and write $\mathbf{P}$ for $\mathbf{P}^0$), and we shall prove that, for $0 < \varepsilon < t$,

$$(17.6)\quad \mathbf{E}[C_t^f - C_\varepsilon^f] = 0.$$

Using the assumption (17.2), the fact that $\mathbf{P}[\sup_{u\leq t}|B_u| \geq a] \leq c\mathbf{P}[|B_1| \geq a/\sqrt{t}]$ (see (13.4)), and dominated convergence, (17.6) implies (17.5).

Letting $p_t(x) := (2\pi t)^{-d/2}\exp(-|x|^2/2t)$ denote the d-dimensional Brownian transition density, we observe that, for $t > 0$, $x \in \mathbb{R}^d$,

$$(17.7)\quad \frac{\partial p_t}{\partial t}(x) = \frac{1}{2}\sum_{j=1}^d \frac{\partial^2 p_t}{\partial x_j^2}(x).$$

Hence

$$\mathbf{E}[C_t^f - C_\varepsilon^f] = E\left[f(t,B_t) - f(\varepsilon,B_\varepsilon) - \int_\varepsilon^t \mathscr{G} f(s,B_s)\,ds\right]$$

$$= \int [p_t(x)f(t,x) - p_\varepsilon(x)f(\varepsilon,x)]\,dx$$

$$- \int_\varepsilon^t ds \int p_s(x)\left[\frac{\partial f}{\partial t}(s,x) + \tfrac{1}{2}\Delta f(s,x)\right]dx.$$

But

$$\int p_s(x)\tfrac{1}{2}\Delta f(s,x)\,dx = \int \tfrac{1}{2}\Delta p_s(x) f(s,x)\,dx,$$

(integrating twice by parts and using (17.2))

$$= \int \frac{\partial p_s}{\partial t}(x) f(s,x)\,dx,$$

using (17.7). Thus

$$\mathbf{E}[C_t^f - C_\varepsilon^f] = \int [p_t(x)f(t,x) - p_\varepsilon(x)f(\varepsilon,x)]\,dx$$

$$- \int_\varepsilon^t ds \int\left[p_s(x)\frac{\partial f}{\partial t}(s,x) + f(s,x)\frac{\partial p_s}{\partial t}(x)\right]dx$$

$$= \int [p_t(x)f(t,x) - p_\varepsilon(x)f(\varepsilon,x)]\,dx - \int_\varepsilon^t ds \int \frac{\partial}{\partial t}(p_s(x)f(s,x))\,dx$$

$$= \int [p_t(x)f(t,x) - p_\varepsilon(x)f(\varepsilon,x)]\,dx - \int\left\{\int_\varepsilon^t ds\,\frac{\partial}{\partial t}[p_s(x)f(s,x)]\right\}dx$$

$$= 0.$$

□