1. GAUSSIAN ISOPERIMETRIC INEQUALITY

Always γ_m denotes the standard Gaussian measure on $\mathbb{R}^m$, or on any vector space with a given inner product (for example, if W is a k-dimensional subspace of $\mathbb{R}^m$, we use γ_k for the Gaussian measure on W with the inherited inner product).

Theorem 1 (Borell, Tsirelson-Ibragimov-Sudakov (1970s)). *Let A be any Borel subset of $\mathbb{R}^m$ with $\gamma_m(A) > 0$ and let H be a half-space in $\mathbb{R}^m$ with $\gamma_m(H) = \gamma_m(A)$. Then $\gamma_m(A^\varepsilon) \geq \gamma_m(H^\varepsilon)$ for all $\varepsilon > 0$. If A is a closed set with $\gamma_m(A) > 0$, then equality holds for some $\varepsilon > 0$ if and only if A is a half-space.*

We present an honest and complete (hopefully!) proof of this theorem[3]. First, the one-dimensional case as an exercise. We give a solution later, since Theorem 1 will be proved by induction on m.

Exercise 2. For any closed set $A \subseteq \mathbb{R}$ and any $\varepsilon > 0$, we have $\Phi^{-1}(\gamma_1(A^\varepsilon)) \geq \Phi^{-1}(\gamma_1(A)) + \varepsilon$. [**Hint:** Try proving it for one interval and then a finite union of intervals. From there to closed sets may be omitted.]

Notation: For a unit vector $\mathbf{u} \in \mathbb{R}^m$ and $t \in \mathbb{R}$, define the closed half-space $H_{\mathbf{u}}(t) := \{\mathbf{x} : \langle \mathbf{x}, \mathbf{u} \rangle \leq t\}$. For a closed subset $A \subseteq \mathbb{R}^m$, define

- $M(A) := \{B \subseteq \mathbb{R}^m : \text{B is closed}, \gamma_m(A) = \gamma_m(B), \gamma_m(A^\varepsilon) \geq \gamma_m(B^\varepsilon) \text{ for all } \varepsilon > 0\}$.
- $r(A) := \inf\{t \in \mathbb{R} : A \subseteq H_{\mathbf{u}}(t) \text{ for some unit vector } \mathbf{u}\}$.

The set $M[A]$ is the collection of all closed sets that are at least as good as A from the isoperimetry point of view. The quantity $r(A)$ will be of use in proofs. We now collect some basic facts about $M[A]$ and $r(A)$.

Lemma 3. *Let $\mathcal{C}$ be the set of all closed subsets of $\mathbb{R}^m$ endowed with the Hausdorff metric d.*

(1) The function $A \to r(A)$ is continuous.

(2) The function $A \to \gamma_m(A)$ is upper semicontinuous.

(3) If A is a closed subset of $\mathbb{R}^m$ with $\gamma_m(A) > 0$, then $r(\cdot)$ attains its minimum on $M(A)$.

The main idea in proving Theorem 1 is a symmetrization procedure due to Antoine Ehrhard (analogous to Steiner's symmetrization for the classical isoperimetric inequality in Euclidean space) that takes a set and produces another that is better in the isoperimetric sense.

Ehrhard's symmetrization: Let ℓ be a one-dimensional affine subspace in $\mathbb{R}^m$ and let $\mathbf{u} \in \ell^\perp$ be a unit vector. For any $A \subseteq \mathbb{R}^m$, define its symmetrization w.r.t. $(\ell, \mathbf{u})$ as the subset $B = S_{\ell,\mathbf{u}}[A]$ such that

(1) for any $t \in \ell$, the section $B \cap (t + \ell^\perp)$ is a half-space in $t + \ell^\perp$ whose boundary is orthogonal to $\mathbf{u}$,

(2) $\gamma_{m-1}(B \cap (t + \ell^\perp)) = \gamma_{m-1}(A \cap (t + \ell^\perp))$.

Here is a more explicit description of B. For each $t \in \mathbb{R}$, find the unique $a = a_t \in \mathbb{R} \cup \{\pm\infty\}$ such that $\gamma_{m-1}(H_{\mathbf{u}}(a) \cap (t + \ell^\perp)) = \gamma_{m-1}(A \cap (t + \ell^\perp))$ and set $B = \bigcup_{t \in \ell}(H_{\mathbf{u}}(a_t) \cap (t + \ell^\perp))$.

As $S_{\ell,\mathbf{u}}[A]$ is defined by an uncountable union of sections, it is not obvious that it is measurable, even for a nice set A. The following lemma shows that any symmetrization transforms closed sets to closed sets, in particular measurable.

Lemma 4. *Let A be a closed set. Then $S_{\ell,\mathbf{v}}[A]$ is also closed.*

[3] Our proof is cobbled together from the paper of Ehrhard, *Symétrisation de l'espace de Gauss* and the appendix to the paper of Figiel, Lindenstrauss and Milman, *The dimension of almost spherical sections of convex bodies*. The symmetrization idea is from Ehrhard. But the rest of the details needed to complete the proof seems most cleanly presented in the paper of Figiel, Lindenstrass and Milman, albeit for the isoperimetric inequality on the sphere. These details appear to go through for the Gaussian case with minimal modification. If there are gaps or mistakes, please let me know.

The following two lemmas show why symmetrization improves a set and that the only sets that cannot be improved by further symmetrizations are half-spaces. They justify the use of symmetrization as a tool and their proofs form the heart of the proof of Theorem 1.

Lemma 5. *Let A be closed and non-empty in $\mathbb{R}^m$. Then $S_{(\ell,\mathbf{v})}[A] \in M(A)$ for any symmetrization $(\ell,\mathbf{v})$.*

Lemma 6. *Let A be a non-empty closed subset of $\mathbb{R}^m$. Then there exist a finite sequence of symmetrizations under which A transforms to a set B with $r(B) < r(A)$.*

Now we prove the main theorem assuming all the lemmas stated so far.

Proof of Theorem 1. Let A be any closed set with $\gamma_m(A) > 0$. By the third part of Lemma 3, there is some $B \in M[A]$ with $r(B) \le r(X)$ for all $X \in M[A]$. If B is not a half-space, then by Lemma 6 we could get apply a finite number of symmetrizations to get a set C with $r(C) < r(B)$. Lemma 5 implies that $C \in M[B]$. But since $M[B] \subseteq M[A]$ this contradicts the minimality of $r(B)$. Thus, B must be a half-space. This proves the isoperimetric inequality for closed sets A. Recall that (2) is an equivalent form of the inequality and thus it has been proved now for closed sets.

If A is any Borel set, by regularity of γ_m, for any $\delta > 0$ there exists a compact sets $K \subseteq A$ with $\gamma_m(K) \ge \gamma_m(A) - \delta$. Then

$$
\begin{aligned}
\Phi^{-1}(\gamma_m(A^\varepsilon)) &\ge \Phi^{-1}(\gamma_m(K^\varepsilon)) \quad \text{(because } K \subseteq A\text{)} \\
&\ge \Phi^{-1}(\gamma_m(K)) + \varepsilon \quad \text{(by the proved inequality (2) for closed sets)} \\
&\ge \Phi^{-1}(\gamma_m(A) - \delta) + \varepsilon. \quad \text{(because } \Phi^{-1} \text{ is increasing)}
\end{aligned}
$$

Let $\delta \downarrow 0$ to get $\Phi^{-1}(\gamma_m(A^\varepsilon)) \ge \Phi^{-1}(\gamma_m(A)) + \varepsilon$. ∎

2. Proofs of lemmas used to prove the Gaussian isoperimetric inequality

Proof of Lemma 3.

(1) Suppose $d(A,B) < \delta$ for some $A, B \in \mathcal{C}$. If a half-space H contains A, then H^δ contains B. Therefore $r(B) \le r(A) + \delta$. Reversing the roles of A and B we see that $A \to r(A)$ is in fact a Lipschitz function on $\mathcal{C}$.

(2) If $d(A,B) < \delta$ then $A^\delta \supseteq B$ and hence $\gamma_m(A^\delta) \ge \gamma(B)$. Hence, if $d(A_k, A) \to 0$, then $\gamma_m(A^\delta) \ge \limsup \gamma_m(A_k)$ as $k \to \infty$. This holds for any δ and $\gamma_m(A^\delta) \to \gamma_m(A)$ as $\delta \to 0$. Therefore $\gamma_m(A^\delta) \ge \limsup_{k\to\infty} \gamma_m(A_k)$ showing that γ_m is u.s.c. on $\mathcal{C}$.

(3) Let $r = \inf\{r(X) : X \in A\}$. Since $\Phi(r(X)) \ge \gamma_m(A) > 0$ for all $X \in M[A]$, it follows that $r > -\infty$. If $r = +\infty$, then we may take $B = A$. Thus we assume that r is finite.

Let $B_k \in M[A]$ with $r_k := r(B_k) \downarrow r$. Then $B_k \subseteq H_{\mathbf{u}_k}(r_k + 1/k)$ for some unit vectors $\mathbf{u}_k$. By passing to a subsequence we may assume that $\mathbf{u}_k \to \mathbf{u}$ for some unit vector $\mathbf{u}$. Since $\gamma_m(B_k) = \gamma_m(A) > 0$, there is a finite number R_0 such that $B(0,R_0)$ has a non-empty intersection with B_k for all k. By Lemma 8, we can pass to a further subsequence and assume that $B_k \cap K \to B \cap K$ in Hausdorff metric for every compact set K. Here B is a closed set.

By the second part, $\gamma_m(B \cap K) \ge \limsup \gamma_m(B_k \cap K) \ge \limsup \gamma_m(B_k) - \gamma_m(K^c)$. Since $B_k \in M[A]$, by taking arbitrarily large K we get $\gamma_m(B) \ge \gamma_m(A)$.

Now fix K. For any $\delta > 0$ we have $B \cap K \subseteq (B_k \cap K)^\delta$ for large enough k and hence $\gamma_m((B\cap K)^\varepsilon) \le \liminf \gamma_m((B_k \cap K)^{\delta+\varepsilon}) \le \gamma_m(A^{\varepsilon+\delta})$ since each $B_k \in M[A]$. Now let $\delta \downarrow 0$ to get $\gamma_m((B\cap K)^\varepsilon) \le \gamma_m(A^\varepsilon)$ for all $\varepsilon > 0$. Then let K increase to $\mathbb{R}^m$ and conclude that $\gamma_m(B^\varepsilon) \le \gamma_m(A^\varepsilon)$. Thus, $B \in M[A]$.

We claim that $B \subseteq H_{\mathbf{u}}(r)$. For if not, then for some small enough $\delta > 0$ and large enough compact set K we must have $(B \cap K) \cap \partial H_{\mathbf{u}}(r+\delta) \ne \emptyset$. But for large enough k we have $B_k \cap K \subseteq H_{\mathbf{u}}(r + \delta/3)$ and $B \cap K \subseteq (B_k \cap K)^{\delta/3}$ which implies that $B \cap K \subseteq H_{\mathbf{u}}(r + 2\delta/3)$, a contradiction.

Putting everything together, we have found a set $B \in M[A]$ and $B \subseteq H_{\mathbf{u}}(r)$. Thus $r(B) = r$ and the proof is complete. ■

Proof of Lemma 4. Fix ℓ and $\mathbf{v}$ and write points of $\mathbb{R}^m$ as $(t, \mathbf{x})$ with $t \in \ell$ and $\mathbf{x} \in \ell^{\perp}$. For any set A, let $A_t = A \cap (t + \ell^{\perp})$ for $t \in \ell$.

Suppose $t_k \to t$. If $(t_k, \mathbf{x}_k) \in A$ and $(t_k, \mathbf{x}_k) \to (t, \mathbf{x})$, then $(t, \mathbf{x}) \in A$ as A is closed. Therefore, $A_t \supseteq \limsup A_{t_k}$, in particular $\gamma_{m-1}(A_t) \geq \limsup \gamma_{m-1}(A_{t_k})$. This implies $a_t \geq \limsup a_{t_k}$.

Now let $B := S_{\ell,\mathbf{v}}[A]$ and suppose that $(t_k, \mathbf{y}_k) \in B$ and $(t_k, \mathbf{y}_k) \to (t, \mathbf{y})$. By definition of symmetrization, $\langle \mathbf{y}_k, \mathbf{v} \rangle \leq a_{t_k}$ and hence $\langle \mathbf{y}, \mathbf{v} \rangle \leq \limsup a_{t_k} \leq a_t$ which implies that $(t, \mathbf{y}) \in B$. Thus B is closed. ■

Proof of Lemma 5. Fix $\ell, \mathbf{v}$ and let $B = S_{(\ell,\mathbf{v})}[A]$. We need to prove two things.

$$(a)\ \gamma_m(B) = \gamma_m(A) \qquad \text{and} \qquad (b)\ \gamma_m(B^{\varepsilon}) \leq \gamma_m(A^{\varepsilon}) \text{ for each } \varepsilon > 0.$$

The first assertion is easy. Use Fubini's theorem to see that

$$\gamma_m(A) = \int_{\mathbb{R}} \gamma_{m-1}[(t\mathbf{u} + \ell^{\perp}) \cap A] d\gamma_1(t) = \int_{\mathbb{R}} \gamma_{m-1}[(t\mathbf{u} + \ell^{\perp}) \cap B] d\gamma_1(t) = \gamma_m(B).$$

The proof of (b) is non-trivial and it is the key step in the entire proof of Theorem 1. By Fubini's theorem, it suffices to show that $\gamma_{m-1}[(B^{\varepsilon})_t] \leq \gamma_{m-1}[(A^{\varepsilon})_t]$ for all $t \in \ell$, where $A_t := A \cap (t + \ell^{\perp})$ is the t-section of A.

Without loss of generality let $\ell = \mathbb{R}e_1$ and $\mathbf{v} = e_2$. For each $s \in \mathbb{R}$, then $B_s = \{(s, u_2, \ldots, u_n) : u_2 \leq a_s\}$ where $\Phi(a_s) = \gamma_{m-1}(A_s)$. Let π denote the orthogonal projection from $\mathbb{R}^m$ onto $\ell^{\perp} = \text{span}\{e_2, \ldots, e_n\}$.

Fix $t \in \mathbb{R}$. Then $(t, \mathbf{x}) \in B^{\varepsilon}$ if and only if there exists s with $|s - t| \leq \varepsilon$ and $\mathbf{y} \in B_s$ with $|\mathbf{y} - \mathbf{x}| \leq \delta_s := \sqrt{\varepsilon^2 - (s-t)^2}$. This means

$$\pi[(B^{\varepsilon})_t] = \bigcup_{s:|s-t|<\varepsilon} (\pi[B_s])^{\delta_s}, \qquad \pi[(A^{\varepsilon})_t] = \bigcup_{s:|s-t|<\varepsilon} (\pi[A_s])^{\delta_s}. \tag{1}$$

In $\mathbb{R}^{n-1}$, $\pi(B_s)$ is a half-space with the same γ_{m-1} measure as $\pi(A_s)$ (by definition of symmetrization). Therefore, inductively assuming the the Gaussian isoperimetric inequality for lower dimensions (the ground case $m = 1$ is checked in Exercise 2), we get $\gamma_{m-1}[(\pi[B_s])^{\delta_s}] \leq \gamma_{m-1}[(\pi[A_s])^{\delta_s}]$ for each s. Therefore, using the second set-identity in (1) we get $\gamma_{m-1}[(\pi[B_s])^{\delta_s}] \leq \gamma_{m-1}[\pi[(A^{\varepsilon})_t]$ for each $s \in [t - \varepsilon, t + \varepsilon]$.

Now note that $(\pi[B_s])^{\delta_s} = \{(u_2, \ldots u_n) : u_2 \leq a_s + \delta_s\}$ are all half-spaces. For any two of them, one contains the other. Hence, their union is an increasing union of a countable number of them. Therefore,

$$\gamma_{m-1}[\pi((B^{\varepsilon})_t)] = \sup_{s:|s-t|\leq\varepsilon} \gamma_{m-1}[(\pi[B_s])^{\delta_s}] \leq \gamma_{m-1}[\pi[(A^{\varepsilon})_t].$$

Equivalently $\gamma_{m-1}[(B^{\varepsilon})_t] \leq \gamma_{m-1}[(A^{\varepsilon})_t]$. By Fubini's theorem, this proves (b). ■

Proof of Lemma 6. Since A is closed, the infimum in the definition of $r(A)$ is a minimum. Let $\mathbf{v}$ be a unit vector such that $H_{\mathbf{v}}(r) \supseteq A$ with $r = r(A)$. Without loss of generality we assume $\mathbf{v} = e_n$. Let $W = re_n + e_n^{\perp}$, the boundary of the half-space $H := H_{e_n}(r)$.

First pick any line ℓ_0 inside W and let $A' = S_{\ell_0,e_n}[A]$. Since A is closed and not the whole half-space, A' is a closed proper subset of H. Further, if $\mathbf{x} \in A'$ and $\mathbf{y} \in H$ has $y_i = x_i$ for $i \leq n-1$ and $y_n < x_n$, then $\mathbf{y} \in A'$ too. Therefore, it is clear that there is a point $p \in W$ and $\delta > 0$ such that $A' \cap Q_p(2\delta) = \emptyset$ where $Q_p(2\delta) = p + (-2\delta, 2\delta)^n$.

Now let $\ell_i = p + re_n + \mathbb{R}e_i$ for $i = 1, 2, \ldots, n-1$. These are lines inside W, passing through p and parallel to the co-ordinate directions.

Let $A'' = S_{\ell_1,p_1}[A']$. For each $t \in [-\delta, \delta]$ the section $(t + \ell_1^{\perp}) \cap A'$ is a subset of $[(t + \ell_1^{\perp}) \cap (H \setminus Q_p(\delta))]$. Therefore, there is some $\delta' > 0$ such that $A'' \cap ([-\delta, \delta] \times \mathbb{R}^{m-1})$ is contained in $H_{\mathbf{v}}(r - \delta')$.

Now symmetrize w.r.t. $(\ell_2, \mathbf{v})$ and let $A''' = S_{\ell,\mathbf{v}}[A'']$. For each $t \in \ell_2$, the section $A'' \cap (t + \ell_2^{\perp})$ is a subset of $H_{\mathbf{v}}(r - \delta')$. Therefore, there is some $\delta'' > 0$ such that $A''' \subseteq H_{\mathbf{v}}(r - \delta'')$. Thus in (at most) three symmetrizations we arrive at a set A''' with $r(A''') < r(A)$. ∎

Solution to Exercise 2. For $p \in (0,1)$ define $Q_p = \Phi^{-1}(1-p)$, the $(1-p)$-quantile. For $x \le Q_p$, define $b_p(x)$ by the equation $\gamma_1[x, b_p(x)] = p$. Let α_p denote the unique x such that $b_p(x) = -x$. Differentiating $p = \int_x^{b_p(x)} \varphi(t)dt$, we get $\varphi(b_p(x))b_p'(x) - \varphi(x) = 0$.

Fix $p \in (0,1)$, $\varepsilon > 0$ and define $h(x) = \gamma_1[x - \varepsilon, b_p(x) + \varepsilon]$ and observe that

$$h'(x) = \varphi(b_p(x)+\varepsilon)b_p'(x) - \varphi(x-\varepsilon) = \varphi(x)\left\{\frac{\varphi(b_p(x)+\varepsilon)}{\varphi(b_p(x))} - \frac{\varphi(x-\varepsilon)}{\varphi(x)}\right\} = \varphi(x)\left\{\frac{\varphi(b_p(x)+\varepsilon)}{\varphi(b_p(x))} - \frac{\varphi(-x+\varepsilon)}{\varphi(-x)}\right\}.$$

Note that $\varphi(u+\varepsilon)/\varphi(u) = e^{-u\varepsilon - \frac{1}{2}\varepsilon^2}$ is decreasing in x. Hence, when $x > \alpha_p(x)$ (which is equivalent to $b_p(x) > -x$), we have $h'(x) < 0$. Thus $h(\alpha_p) > h(x) > h(Q_p)$ for all $x \in (\alpha_p, Q_p)$.

Case of one closed interval: If A is an interval with $\gamma_1(x) = p$, then it is of the form $[x, b_p(x)]$ for some x. We may also assume that $x \ge \alpha_p$ (otherwise replace A by $-A$). Thus, by the above deduction, $\gamma(A^{\varepsilon})$ is minimized when $x = Q_p$.

Case of multiple closed intervals: We write A as $I_1 \sqcup I_2 \ldots \sqcup I_k$ with $I_j = [x_j, b_p(x_j)]$ with $b_p(x_{i-1}) < x_i$ for all i. There are two reductions which improve our set in isoperimetric setting.

(1) Suppose that I_k and I_{k-1} differ by less than 2ε, i.e., $b_p(x_{k-1}) + \varepsilon > x_k - \varepsilon$. In this case, if we move the interval $[x_k, b_p(x_k)]$ to the left (i.e., decrease x_k), then $\gamma_1(A)$ stays the same but $\gamma_1(A^{\varepsilon})$ decreases till x_k hits $b_p(x_{k-1})$. This results in a set with $(k-1)$ intervals and better isoperimetric profile.

(2) Suppose that I_{k-1} and I_k are separated by at least 2ε. Without loss of generality $b_p(x_k) > -x_k$ (otherwise, replace I_k by $-I_k$, which would be even further to the right than I_k and the separation condition continues to hold). Then, by the earlier deduction, as x_k increases, $\gamma_1(A)$ stays the same but $\gamma_1(A^{\varepsilon})$ decreases, till $x_k = Q_{p_k}$.

Repeatedly applying these two reductions, we can reduce A to the interval $[Q_p, \infty)$.

Case of an arbitrary closed set: Let A be closed with $\gamma_1(A) = p$. For any small $\eta > 0$, the set A^{η} is the closure of an open set, and hence it is a union of countably many disjoint closed intervals. At the cost of losing an η probability, we drop all but finitely many intervals. This gives us a set B with the property that $B \subseteq A^{\eta}$ and $p' := \gamma_1(B) \ge p - \eta$. By the already proved inequality, $\gamma_1(B^{\varepsilon}) \ge \gamma_1[Q_{p'} - \varepsilon, \infty)$. Of course $B^{\varepsilon} \subseteq A^{\eta+\varepsilon}$ and therefore $\gamma_1(A^{\varepsilon}) \ge \gamma_1[Q_{p'} - \varepsilon, \infty)$. Letting $\eta \downarrow 0$ and noticing that $p' \to p$, we get $\gamma_1(A^{\varepsilon}) \ge \gamma_1[Q_p - \varepsilon, \infty)$. ∎

3. Appendix: Hausdroff metric

Let (X,d) be a metric space and let $\mathcal{C}_X$ denote the set of all non-empty closed subsets of X. The Hausdorff distance between two closed sets A, B is defined by $d_H(A,B) = \inf\{r > 0 : A^r \supseteq B \text{ and } B^r \supseteq A\}$ where $A^r = \{\mathbf{x} : d(\mathbf{x}, A) \le r\}$. The value $+\infty$ is allowed and $(\mathcal{C}, d)$ is a metric space (if you are not comfortable with a metric that takes infinite values, just use $d_H(A,B) \wedge 1$ which is a finite metric).

Exercise 7. Let (X,d) be a compact metric space. Then $(\mathcal{C}_X, d_H)$ is a compact metric space.

We shall work in $\mathbb{R}^m$ which is not compact.

Lemma 8. *Let A_k be a sequence of closed non-empty sets in $\mathbb{R}^m$. Assume that $A_k \cap B(0,R_0) \neq \emptyset$ for all k for some R_0. Then, there exists a subsequence k_j and a non-empty closed set X such that $A_{k_j} \cap K \to X \cap K$ in Hausdorff metric for every non-empty compact $K \subseteq \mathbb{R}^m$.*

Proof. For each $j > R_0$, use Exercise 7 to see that $A_k \cap \overline{B(0,j)}$ has a subsequence that converges in Hausdorff metric to some set $X_j \subseteq \overline{B(0,j)}$. Set $X = \cup_j X_j$. Then it is easy to see that X is closed and the conclusions hold (check!). ∎

4. APPENDIX: GAP IN THE PROOF!

In lecture we realized that there is a gap in the proof that we gave for the isoperimetric inequality. It is in the proof of Lemma 6. The given proof is correct in dimensions 3 and higher but not in dimension 2 as there is only one line contained in the boundary of a half space in $\mathbb{R}^2$! We fix this below[4].

Lemma 9. *Let $\mathbf{v}_k = (\cos\theta_k, \sin\theta_k)$ with $\theta_0 = 0$ and $\theta_k = \frac{\pi+\theta_{k-1}}{2}$ for $k \geq 1$. Let $\ell_k = \mathbf{v}_k^\perp$ and let $S_k := S_{\ell_k, -\mathbf{v}_k}$. Given a closed set $A \subseteq \mathbb{R}^2$, define $A_0 = S_0[A]$ and $A_k = S_k[A_{k-1}]$ for $k \geq 1$.*

(1) If $\mathbf{x} \in A_k$ then $\mathbf{x} + t\mathbf{v}_0 + s\mathbf{v}_k \in A_k$ for all $t, s \geq 0$.

(2) Let $H = \{(x,y) : y \geq \Phi^{-1}(\gamma_2(A))\}$. Then A_k converges to H on compacta in Hausdorff metric i.e., $A_k \cap K \to H \cap K$ in Hausdorff metric for every compact set K.

Proof. (1) By definition of symmetrization, it is clear that if $\mathbf{x} \in A_k$ then $\mathbf{x} + t\mathbf{v}_k \in A_k$ for $t > 0$. It remains to prove for $k \geq 1$ that if $\mathbf{x} \in A_k$ then $\mathbf{x} + t\mathbf{v}_0 \in A_k$. The case $k = 0$ is trivial.

Consider $k = 1$. By the $\gamma_1(A \cap (t\mathbf{v}_0 + \ell_1^\perp))$ is increasing in t (because of the case $k = 0$), which shows that if $\mathbf{x} \in A_1$ then $\mathbf{x} + t\mathbf{v}_0 \in A_1$. This completes the proof for $k = 1$.

Fix $k \geq 2$ and let π denote the projection onto $\ell_k^\perp$ and let $A_{k-1,t} = A_{k-1} \cap (t\mathbf{v}_k^\perp + \ell_k^\perp)]$ and $A_{k,t} = A_k \cap (t\mathbf{v}_k^\perp + \ell_k^\perp)]$ so that $A_{k,t}$ is a half-line with $\gamma_1(A_{k,t}) = \gamma_1(A_{k-1,t})$. Observe that ℓ_k is the angle bisector of $\mathbf{v}_k$ and $\mathbf{v}_0$. Therefore, inductively assuming the lemma for $k-1$, we see that $\pi[A_{k-1,t+\varepsilon}] \supseteq \pi[A_{k-1,t}]^\varepsilon$ (the ε-enlargement in $\ell^\perp = \mathbb{R}$). Consequently, by the one-dimensional isoperimetric inequality we deduce that $\pi[A_{k,t+\varepsilon}] \supseteq \pi[A_{k,t}]^\varepsilon$. Draw a picture to see that this precisely implies that if $\mathbf{x} \in A_k$ then $\mathbf{x} + t\mathbf{v}_0 \in A_k$ for $t \geq 0$.

(2) If $\gamma_2(A) = 0$ then A_k is empty for all k and the statement is valid. Hence assume $\gamma_2(A) > 0$. By properties of symmetrization, for every k we have $\gamma_2(A_k) = \gamma_2(A)$ and $\gamma_2(A_k^\varepsilon) \leq \gamma_2(A^\varepsilon)$ for all $\varepsilon > 0$. Also define the cone $\mathcal{C}_k = \{s\mathbf{v}_0 + t\mathbf{v}_k : s, t \geq 0\}$ and $\mathcal{C}_\infty = \{(x,y) : y \geq 0\}s$.

Let R be large enough such that $\gamma_2(B_0(R)^c) < \gamma_2(A)$. Then there exists $\mathbf{x}_k \in A_k \cap B_0(R)$. Having fixed $\varepsilon > 0$ and $R > 0$, it is clear that for large enough k and every $\mathbf{x} \in B_0(R)$ we have $((\mathbf{x} + \mathcal{C}_k) \cap B_0(R))^\varepsilon \supseteq (\mathbf{x} + \mathcal{C}_\infty) \cap B_0(R)$. Since $\mathcal{C}_k \subseteq \mathcal{C}_\infty$ we obviously have $((\mathbf{x} + \mathcal{C}_\infty) \cap B_0(R))^\varepsilon \supseteq (\mathbf{x} + \mathcal{C}_k) \cap B_0(R)$. ■

[4] Proof is taken from Bogachev's book, chapter 4