Hypercontractivity:

Thm 4.12: Let $A: H_1 \to H_2$ be a linear operator between two Gaussian Hilbert Spaces, defined on probability spaces $(\Omega_i, \mathcal{F}_i, P_i)$ such that $\|A\| \le 1$. Then $\Gamma(A)$ can be (uniquely) extented to a continuous linear operator $L^1(\Omega_1, \mathcal{F}(H_1), P_1) \to L^1(\Omega_2, \mathcal{F}(H_2), P_2)$, which we also denote by $\Gamma(A)$. Furthermore,

i) $\|\Gamma(A)X\|_p \le \|X\|_p$ for any $X \in L^p$, $1 \le p \le \infty$;

ii) if $X \ge 0$ a.s., then $\Gamma(A)X \ge 0$ a.s.

iii) $|\Gamma(A)X| \le \Gamma(A)|X|$ a.s. for any $X \in L^1$.

Theorem 2. Let H_1, and H_2 be Gaussian Hilbert spaces defined on probability spaces $(\Omega_i, \mathcal{F}(H_i), P_2)$. If $1 \le p \le q \le \infty$ and $A: H_1 \to H_2$ is a linear map with norm $\|A\| \le \left(\frac{p-1}{q-1}\right)^{1/2}$, then $\Gamma(A)$ maps $L^p(\Omega_1, \mathcal{F}(H_1), P_1)$ into $L^q(\Omega_2, \mathcal{F}(H_2), P_2)$ with norm $\|\Gamma(A)\|_{p,q} = 1$.

Remark: The condition $\cancel{\|A\|}$ is sharp; if $\|A\| > \sqrt{(p-1)/(q-1)}$, then $\Gamma(A)$ does not even map L^p into L^q.

Reason: if $H_1 = H_2 \to$ one dimensional space. and $r^2 > (p-1)/(q-1)$.

Then $\Gamma(rI)(e^{ax^2}) \notin L^q$ for some a s.t. $e^{ax^2} \in L^p$.

Note: $\cancel{e^{ax^2}} = \sum_{k=0}^{\infty} \cancel{\frac{a^k}{(1-2a)^{k+1/2} k!} h_{2k}(x)} \Rightarrow \cancel{\Gamma(rI)(e^{ax^2}) = e^{ar^2x^2}}$

Proof of thm2: We have $\Gamma(A) = \Gamma(\|A\| I)\,\Gamma(A/\|A\|)$. Hence it is suffices to consider the case $H_1 = H_2$ and $A = rI$ with $0 \le r \le \sqrt{\left(\frac{p-1}{q-1}\right)}$.

Reason: $$\Gamma(H_1) \xrightarrow{\Gamma(A/\|A\|)} \Gamma(H_2) \xrightarrow{\Gamma(\|A\| I)} \Gamma(H_2)$$

By thm 1. we have

$$\|\Gamma(A)X\|_q = \|\Gamma(\|A\| I)\left(\Gamma(A/\|A\|)X\right)\|_q \overset{\boxed{\text{show.}}\ ??}{\le} \|\Gamma(A/\|A\|)X\|_p$$

$$\overset{\text{thm 1}}{\le} \|X\|_p.$$

Claim: $\|\Gamma(rI)X\|_q \le \|X\|_p$ for $X \in L^p(\Omega_1, \mathcal{F}(H_1), P_1)$.

$\mathcal{X} = \{x_1, \ldots, x_m\}$ → set of r.v.s.

- Definition: $T_{\mathcal{X},r}\left(\sum_n Y_n\right) = \sum_n r^n Y_n$, $Y_n \in \mathcal{P}_n(\mathcal{X}) \cap \mathcal{P}_{n-1}(\mathcal{X})^{\perp}$

If $Y \in \mathcal{P}(\mathcal{X})$ → space of all polynomial generated by $\mathcal{X} = \{x_1, \ldots, x_m\}$.

$Y = \sum_n Y_n$

We say that the set $\{x_1, \ldots, x_m\}$ is (p,q,r)-hypercontractive, if

$T_{\mathcal{X},r} : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$.

$$\|T_{\mathcal{X},r}(Y)\|_q \le \|Y\|_p, \quad Y \in \mathcal{P}(\mathcal{X}).$$

- Note, if the ~~$\mathcal{X}$ the va~~ variables in $\mathcal{X}$ have a Gaussian distribution, then $\mathcal{P}$ and $\mathcal{P}_n$ are as defined in Ch. 2 for Gaussian Hilbert space spanned by $\mathcal{X}$ and $T_{\mathcal{X},r} = \Gamma(rI)$.

Hence we want to show any such set is (p,q,r)-hypercontractive.

②

<u>Lemma 1.</u> Let X be a random variable with the symmetric two point distribution $P(X=1)=P(X=-1)=\frac{1}{2}$. If $r^2 \le \sqrt{\frac{p-1}{q-1}}$, Then X is (p,q,r) hypercontractive.

<u>Lemma 2:</u> Suppose that $X=\{x_1,\dots,x_m\}$ and $Y=\{y_1,\dots,y_m\}$ are two independent (p,q,r)-hypercontractive sets of random variables. Then their union $\{x_1,\dots,x_m,y_1,\dots,y_n\}$ is (p,q,r)-hypercontractive.

<u>Proof of claim 1.</u> Fix, p,q, and r with $r^2 \le \frac{p-1}{q-1}$.

Let, $x_1,\dots,x_m$ be independent random variables with the symmetric distribution as in Lemma 1. And define $S_m=\sum_{i=1}^{m} X_i$. By Lemma 1 & Lemma 2. $\{x_1,\dots,x_m\}$ - (p,q,r) hypercontractive.

<u>Claim 1.</u> i) S_m is (p,q,r)-hypercontractive.

ii) $\frac{S_m}{\sqrt{m}}$ is (p,q,r) " .

iii) ξ is (p,q,r) " , $\xi \sim N(0,1)$.

iv) $\{\xi_1,\xi_2,\dots,\xi_m\}$ (p,q,r) "

v) $X \in P_n(H)$, $X=f(\xi_1,\dots,\xi_n)$ for some polynomial f and Gaussian variables $\xi_i \in H$, and $\Gamma(rI)X = T_{\{\xi_1,\dots,\xi_n\},r}(X)$

Hence, $\|\Gamma(rI)X\|_q \le \|X\|_p$.

Since, the set of such variables dense in L^p.

Hence, $\|\Gamma(rI)X\|_q \le \|X\|_p$ $\forall X \in L^p(\Omega, \mathcal{F}(H), P)$

Hence, the claim 1.

Proofs i) It is clear that $\mathcal{P}(S_m) \subseteq \mathcal{P}(x_1, \ldots, x_m)$.

$\mathcal{P}_n(S_m) \to$ consists of all symmetric polynomials in $\mathcal{P}_n(x_1, \ldots, x_m)$

Thus if $X \in \mathcal{P}_n(S_m)$ & $X \in \mathcal{P}_n(S_m)^{\perp} \Rightarrow X \in \mathcal{P}_n\{x_1, \ldots, x_m\}^{\perp}$

i.e. $\mathcal{P}_n(S_m) \cap \mathcal{P}_{n-1}(S_m)^{\perp} \subseteq \mathcal{P}_n\{x_1, \ldots, x_m\} \cap \mathcal{P}_{n-1}\{x_1, \ldots, x_m\}^{\perp}$

and $T_{S_m, r}(Y) = T_{\{x_1, \ldots, x_m\}, r}(Y) \quad \forall\, Y \in \mathcal{P}(S_m)$.

$\Rightarrow S_m$ is (p, q, r) - hyper contractive.

ii) As, consequence, $\tilde{S}_m = \frac{S_m}{\sqrt{m}}$ is (p, q, r) - hyper contractive.

iii) Now, $T_{\tilde{S}_m, r}\left(\sum_{k=0}^{m} a_k P_{m,k}(\tilde{S}_m)\right) = \sum_{k=0}^{m} a_k r^k P_{m,k}(\tilde{S}_m)$

$P_{m,k}$, $k = 0, \ldots, m$ are orthogonal polynomials for the distribution of $\tilde{S}_m$. upto change of variable, these are orthogonal polynomial w.r. to B.D.

Krawtchouk polyn.

Now let $m \to \infty$. $\tilde{S}_m \to \xi$ $(N(0,1))$ [by CLT]

$P_{m,k}(x) \to h_k(x)$

$\hookrightarrow$ Hermit polynomials.

- It follows that for any finite sequence $a_0, \ldots, a_l$ holds

$$\sum_{k=1}^{l} a_k P_{m,k}(\tilde{S}_m) \to \sum_{k=1}^{l} a_k h_k(\xi)$$ in distribution and in every L^p.

consequently, using (p, q, r) - hypercontractivity of each $\tilde{S}_m$,

$$\left\| T_{\xi, r}\left(\sum_k a_k h_k(\xi)\right)\right\|_q = \left\| \sum_{k} a_k r^k h_k(\xi)\right\|_q$$

$$= \lim_{m \to \infty} \left\| \sum_k a_k r^k P_{m,k}(\tilde{S}_m)\right\|_q$$

$$= \lim_{m \to \infty} \left\| \sum_k a_k T_{\tilde{S}_m, r}(P_{m,k}(\tilde{S}_m))\right\|_q$$

$$\leq \lim_{m \to \infty} \left\| \sum_k a_k P_{m,k}(\tilde{S}_m)\right\|_p$$

$$= \left\| \sum_k a_k h_k(\xi)\right\|_p$$

$\Rightarrow \xi$ is (p, q, r) hyper contractivity.

Proof of Lemma 1.

The space $\mathcal{P}(X)$ of polynomial variables is two dimensional and spanned by 1 and X, with $T_{X,r} : a+bX \to a+brX$.

Hence, claim can be written

$$\left(\frac{|a+rb|^q+|a-rb|^q}{2}\right)^{1/q} \le \left(\frac{|a+b|^p+|a-b|^p}{2}\right)^{1/p} \quad \text{---} (*)$$

(this is known as two point inequality)

Note: $U: a+bX \to a-bX$, then $T_{X,r} = \frac{1}{2}\left((1+r)+(1-r)U\right)$

$$\Rightarrow |T_{X,r}(f(x))| \le \frac{1+r}{2}|f(x)| + \frac{1-r}{2}|U(f(x))| = T_{X,r}(|f(x)|).$$

Hence it is suffices to consider the case when $a+bX \ge 0$ ie $a \ge 0$ and $-a \le b \le a$.

In (*) we ~~can~~ ~~will~~ ~~get~~ may assume $a=1$, $|b|<1$.

$$\therefore \left(\frac{(1+rb)^q+(1-rb)^q}{2}\right)^{1/q} \le \left(\frac{(1+b)^p+(1-b)^p}{2}\right)^{1/p}$$

$$\Rightarrow \left(\sum_{k=0}^{\infty}\binom{q}{2k} r^{2k} b^{2k}\right)^{p/q} \le \left(\sum_{k=0}^{\infty}\binom{p}{2k} b^{2k}\right)$$

For $1 \le p \le q \le 2$.

$$\text{LHS} = \left(1+\sum_{k=1}^{\infty}\binom{q}{2k} r^{2k} b^{2k}\right)^{p/q}$$

$$\le 1 + \frac{p}{q}\sum_{k=1}^{\infty}\binom{q}{2k} r^{2k} b^{2k}$$

$\left[\because (1+x)^{\lambda} \le 1+\lambda x \text{ for } x \ge 0,\ 0 \le \lambda \le 1\right]$

$$\le 1 + \sum_{k=1}^{\infty}\binom{p}{2k} b^{2k}$$

$$= \sum_{k=0}^{\infty}\binom{p}{2k} b^{2k}.$$

$\because \frac{p}{q}\binom{q}{2k} r^{2k} \le \binom{p}{2k}$

$$\frac{p}{q}\cdot\frac{q(q-1)\cdots(q-k+1)}{p(p-1)\cdots(p-k+1)}\cdot\frac{(p-1)^k}{(q-1)^k}$$

$$\le \frac{(q-2)\cdots(q-2k+1)}{(p-2)\cdots(p-2k+1)} \le 1.$$

1 p q 2

$\therefore$ Lemma 1 is true for $1 \le p \le q \le 2$. By duality we obtain the result for $2 \le p \le q \le \infty$.

for $p \le 2 \le q$ follows from the $p \le q = 2$ and $2 = p \le q$

Proof of lemma 2:

Any polynomial variable in $P(x \cup y)$ can be written as a finite sum $\sum_i f_i(x) g_i(y)$, where f_i, g_i are polynomials, and

$$T_{x\cup y, r}\left(\sum_i f_i(x) g_i(y)\right) = \sum_i \left(T_{x,r} f_i(x)\right)\left(T_{y,r}\, g_i(y)\right)$$

Let, μ_x and μ_y denote distribution of x and y.

Now,

$$\left\| \sum_i T_{x,r} f_i(x) T_{y,r}\, g_i(y) \right\|_q = \left\| \left\| T_{y,r}\left(\sum_i T_{x,r} f_i(x) g_i(y)\right) \right\|_{q(\mu(y))} \right\|_{L^q(\mu(x))}$$

$$\overset{\text{contractivity of } T_{y,r}}{\leq} \left\| \left\| \sum_i T_{x,r} f_i(x) g_i(y) \right\|_{L^p(\mu(y))} \right\|_{L^q(\mu(x))}$$

$$\overset{C.4}{\leq} \left\| \left\| \sum_i T_{x,r} f_i(x) g_i(y) \right\|_{L^q(\mu(x))} \right\|_{L^p(\mu(y))}$$

$$\overset{\text{contractivity of } T_{x,r}}{\leq} \left\| \left\| \sum_i f_i(x) g_i(y) \right\|_{L^p(\mu(x))} \right\|_{L^p(\mu(y))}$$

$$= \left\| \sum_i f_i(x) g_i(y) \right\|_{L^p}$$

(C.4) Let, $(M_1, \mathcal{M}_1, \mu_1)$ and $(M_2, \mathcal{M}_2, \mu_2)$ be two σ-finite measure spaces. If f is a measurable function on $M_1 \times M_2$ and $0 < p \leq q \leq \infty$, then

$$\left\| \| f \|_{L^p(M_1, \mu_1)} \right\|_{L^q(M_2, \mu_2)} \leq \left\| \| f \|_{L^q(M_2, \mu_2)} \right\|_{L^p(M_1, \mu_1)}$$