Homework 1: Due 19th Aug

Submit the first four problems only

1. Let $\mathcal{F}$ be a σ-algebra of subsets of Ω.

(1) Show that $\mathcal{F}$ is closed under countable intersections ($\bigcap_n A_n$), under set differences ($A \setminus B$), under symmetric differences ($A \Delta B$).

(2) If A_n is a countable sequence of subsets of Ω, the set $\limsup_n A_n$ (respectively $\liminf_n A_n$) is defined as the set of all $\omega \in \Omega$ that belongs to infinitely many (respectively, all but finitely many) of the sets A_n.

If $A_n \in \mathcal{F}$ for all n, show that $\limsup A_n \in \mathcal{F}$ and $\liminf A_n \in \mathcal{F}$. [**Hint:** First express $\limsup A_n$ and $\liminf A_n$ in terms of A_ns and basic set operations].

(3) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, what are $\limsup A_n$ and $\liminf A_n$?

2. Let $(\Omega, \mathcal{F})$ be a set with a σ-algebra.

(1) Suppose $\mathbf{P}$ is a probability measure on $\mathcal{F}$. If $A_n \in \mathcal{F}$ and A_n increase to A (respectively, decrease to A), show that $\mathbf{P}(A_n)$ increases to (respectively, decreases to) $\mathbf{P}(A)$.

(2) Suppose $\mathbf{P} : \mathcal{F} \to [0, 1]$ is a function such that (a) $\mathbf{P}(\Omega) = 1$, (b) $\mathbf{P}$ is finitely additive, (c) if $A_n, A \in \mathcal{F}$ and A_ns increase to A, then $\mathbf{P}(A_n) \uparrow \mathbf{P}(A)$. Then, show that $\mathbf{P}$ is a probability measure on $\mathcal{F}$.

3. (1) Let X be an arbitrary set. Let S be the collection of all singletons in Ω. Describe $\sigma(S)$.

(2) Let $S = \{(a, b] \cup [-b, -a) : a < b \text{ are real numbers}\}$. Show that $\sigma(S)$ is strictly smaller than the Borel σ-algebra of $\mathbb{R}$.

(3) Suppose S is a collection of subsets of X and a, b are two elements of X such that any set in S either contains a and b both, or contains neither. Let $\mathcal{F} = \sigma(S)$. Show that any set in $\mathcal{F}$ has the same property (either contains both a and b or contains neither).

4. Let Ω be an infinite set and let $\mathcal{A} = \{A \subseteq \Omega : A \text{ is finite or } A^c \text{ is finite }\}$. Define $\mu : \mathcal{A} \to \mathbb{R}_+$ by $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A^c is finite.

(1) Show that $\mathcal{A}$ is an algebra and that μ is finitely additive on $\mathcal{A}$.

(2) Under what conditions does μ extend to a probability measure on $\mathcal{F} = \sigma(\mathcal{A})$?

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Do not submit the following problems, but at least read them. Some of them are straightforward exercises, but some are more advanced material which I don't get time to elaborate in class. The latter are meant for the extra-curious!

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5. (1) Let $\mathcal{B}$ be the Borel sigma-algebra of $\mathbb{R}$. Show that $\mathcal{B}$ contains all closed sets, all compact sets, all intervals of the form $(a,b]$ and $[a,b)$.

(2) Show that there is a countable family $\mathcal{S}$ of subsets of $\mathbb{R}$ such that $\sigma(\mathcal{S}) = \mathcal{B}_{\mathbb{R}}$.

6. Let $A_1, A_2, \ldots$ be a finite or countable partition of a non-empty set Ω (i.e., A_i are pairwise disjoint and their union is Ω). What is the σ-algebra generated by the collection of subsets $\{A_n\}$? What is the algebra generated by the same collection of subsets?

7. Let $X = [0,1]^{\mathbb{N}}$ be the countable product of copies of $[0,1]$. We define two sigma algebras of subsets of X. Then $d(x,y) = \sum_n |x_n - y_n| 2^{-n}$ defines a metric on X. Let $\mathcal{B}_X$ be the Borel sigma-algebra of (X,d).

Let $\mathcal{C}_X$ be the collection of all cylinder sets, i.e., sets of the form $A = U_1 \times U_2 \times \ldots$ where U_i is a Borel subset of $[0,1]$ for each i and $U_i = [0,1]$ for all but finitely many i.

(1) Show that $\mathcal{C}_X$ is a π-system.

(2) Show that $\sigma(\mathcal{C}_X) = \mathcal{B}_X$.

8. Let μ be the Lebesgue p.m. on the Cartheodary σ-algebra $\overline{\mathcal{B}}$ and let μ_* be the corresponding outer Lebesgue measure defined on all subsets of $[0,1]$. We say that a subset $N \subseteq [0,1]$ is a null set if $\mu_*(N) = 0$. Show that

$$\overline{\mathcal{B}} = \{B \cup N : B \in \mathcal{B} \text{ and } N \text{ is null}\}$$

where $\mathcal{B}$ is the Borel σ-algebra of $[0,1]$.

[**Note:** The point of this exercise is to show how much larger is the Lebesgue σ-algebra than the Borel σ-algebra. The answer is, not much. Up to a null set, every Lebesgue measurable set is a Borel set. However, cardinality-wise, there is a difference. The Lebesgue σ-algebra is in bijection with $2^{\mathbb{R}}$ while the Borel σ-algebra is in bijection with $\mathbb{R}$.]