Homework 2: Due 31st Aug
Submit the first four problems only

1. Suppose X_n are real-valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that $X(\omega) := \lim_{n\to\infty} X_n(\omega)$ exists for each $\omega \in \Omega$. Show that X is a random variable (just use definitions, do not invoke any results already stated).

2. For each of the following distributions, find an explicit mapping $T : (0,1) \to \mathbb{R}$ such that $\lambda \circ T^{-1}$ is equal to the given distribution. (1) Cauchy distribution (density is $\frac{1}{\pi(1+x^2)}$). (2) Exponential distribution (density is e^{-x} for $x > 0$). (3) Laplace distribution (density if $e^{-|x|}$ for all $x \in \mathbb{R}$). (4) A beta density $\frac{1}{\pi\sqrt{x(1-x)}}$ for $x \in (0,1)$.

3. (1) Given a CDF $F : \mathbb{R} \to [0,1]$ show that there is a unique way to write it as $F = \alpha F_d + (1-\alpha)F_c$ where $0 \le \alpha \le 1$, F_d, F_c are CDFs, F_c is continuous and F_d increases only in jumps (the last point means that if $[a,b]$ contains no discontinuities of F_d, then $F_d(b) = F_d(a)$).

(2) If μ, μ_c, μ_d are the corresponding probability measures, then show that $\mu(A) = \alpha\mu_d(A) + (1-\alpha)\mu_c(A)$ for all $A \in \mathcal{B}_{\mathbb{R}}$.

4. Let $\mathcal{G}$ be the countable-cocountable sigma algebra on $\mathbb{R}$. Define the probability measure μ on $\mathcal{G}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A^c is countable. Show that μ is *not* the push-forward of Lebesgue measure on $[0,1]$, i.e., there does not exist a measurable function $T : [0,1] \mapsto \Omega$ (w.r.t. the σ-algebras $\mathcal{B}$ and $\mathcal{G}$) such that $\mu = \lambda \circ T^{-1}$.

————————xxxxxxxxxxxxxxxxxxxx————————

Do not submit the following problems, but at least read them. Some of them are straightforward exercises, but some are more advanced material which I don't get time to elaborate in class. The latter are meant for the extra-curious!

————————xxxxxxxxxxxxxxxxxxxx————————

5. Let F_1, F_2 be CDFs on $\mathbb{R}$. Show that the following are also CDFs.

(1) $F : \mathbb{R}^2 \to \mathbb{R}$ defined as $F(x,y) = F_1(x)F_2(y)$.

(2) $F : \mathbb{R}^2 \to \mathbb{R}$ defined as $F(x,y) = F_1(\min\{x,y\})$.

(3) $F : \mathbb{R} \to \mathbb{R}$ defined as $F(x) = F_1(x^3)$.

6. (1) Let $X : \mathbb{R}^n \to \mathbb{R}^m$. Show that X is Borel measurable if it is (a) right continuous or (b) lower semicontinuous or (c) non-decreasing (take $m = n = 1$ for the last one).

(2) If μ is a Borel p.m. on $\mathbb{R}$ with CDF F, then find the push-forward of μ under F.

7. Let $\Omega = X = \mathbb{R}$ and let $T : \Omega \to X$ be defined by $T(x) = x$. We give a pair of σ-algebras, $\mathcal{F}$ on Ω and $\mathcal{G}$ on X by taking $\mathcal{F}$ and $\mathcal{G}$ to be one of $2^{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{R}}$ or $\{\emptyset, \mathbb{R}\}$. Decide for each of the nine pairs, whether T is measurable or not.

8. Given $X : \Omega \to \mathbb{R}^d$, let $\sigma(X)$ denote the smallest sigma algebra on Ω so that X becomes measurable (as always, take Borel sigma algebra on $\mathbb{R}^d$).

(1) Let $X = (X_1, \ldots, X_n)$. Show that X is an $\mathbb{R}^d$-valued r.v. if and only if $X_1, \ldots, X_n$ are (real-valued) random variables. How does $\sigma(X)$ relate to $\sigma(X_1), \ldots, \sigma(X_n)$?

(2) Let $X : \Omega_1 \to \Omega_2$ be a random variable. If $X(\omega) = X(\omega')$ for some $\omega, \omega' \in \Omega_1$, show that there is no set $A \in \sigma(X)$ such that $\omega \in A$ and $\omega' \notin A$ or vice versa. [**Extra!** If $Y : \Omega_1 \to \Omega_2$ is another r.v. which is measurable w.r.t. $\sigma(X)$ on Ω_1, then show that Y is a function of X].