Homework 3: Due 18th Sep

Submit the first four problems only

1. Suppose $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$ and that their distribution functions are continuous. If $\mu_n \xrightarrow{d} \mu$, show that $F_{\mu_n}(t) - F_\mu(t) \to 0$ uniformly over $t \in \mathbb{R}$.

2. Suppose μ_n, μ are discrete probability measures supported on $\mathbb{Z}$ having probability mass functions $(p_n(k))_{k\in\mathbb{Z}}$ and $(p(k))_{k\in\mathbb{Z}}$. Show that $\mu_n \xrightarrow{d} \mu$ if and only if $p_n(k) \to p(k)$ for each $k \in \mathbb{Z}$.

3. Let X be a random variable with distribution μ and let μ_n be the distribution of X_n which is defined as below. In each case, show that $\mu_n \xrightarrow{d} \mu$ as $n \to \infty$.

(1) (Truncation). $X_n = (X \wedge n) \vee (-n)$.

(2) (Discretization). $X_n = \frac{1}{n}\lfloor nX \rfloor$.

4. (1) Show that the family of exponential distributions $\{\text{Exp}(\lambda) : \lambda > 0\}$ is not tight.

(2) For what $A \subseteq \mathbb{R}$ is the restricted family $\{\text{Exp}(\lambda) : \lambda > 0\}$ tight?

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5. Show that under the Lévy metric, $\mathcal{P}(\mathbb{R})$ is a complete and seperable metric space.

6. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$. Show that the following statements are equivalent to $\mu_n \xrightarrow{d} \mu$.

(1) $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ if F is closed.

(2) $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ if G is open.

(3) $\limsup_{n\to\infty} \mu_n(A) = \mu(A)$ if $A \in \mathcal{B}_\mathbb{R}$ and $\mu(\partial A) = 0$.

[Remark: One approach is to first show the second statement for an open interval.]

7. Recall the Cantor set $C = \bigcap_n K_n$ where $K_0 = [0,1]$, $K_1 = [0,1/3] \cup [2/3,1]$, etc. In general, K_n is of the form $\bigcup_{1\leq j\leq 2^n} [a_{n,j}, b_{n,j}]$ where $b_{n,j} - a_{n,j} = 3^{-n}$ for each j.

(1) Let μ_n be the uniform probability measure on K_n. Describe its CDF F_n.

(2) Show that F_n converges uniformly to a CDF F.

(3) Let μ be the probability measure with CDF equal to F. Show that $\mu(C) = 1$.