Some problems for practise

Do not submit

1. (1) Let $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\alpha', 1)$ be independent random variables on a common probability space. Find the distribution of $\frac{X}{X+Y}$.

(2) If U, V are independent and have uniform([0,1]) distribution, find the distribution of $U + V$.

2. (1) Suppose (X, Y) has a continuous density $f(x, y)$. Find the density of X/Y. Apply to the case when (X, Y) has the *standard bivariate normal distribution* with density $f(x, y) = (2\pi)^{-1} \exp\{-\frac{x^2+y^2}{2}\}$.

(2) Find the distribution of $X + Y$ if (X, Y) has the standard bivariate normal distribution.

(3) Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Find the density of (U, V).

3. (1) Let X, Y be independent with $X \sim$ Poisson(λ) and $Y \sim$ Poisson(μ). Show that $X + Y \sim$ Pois$(\lambda + \mu)$.

(2) Let X, Y be independent with $X \sim$ Bin(n, p) and $Y \sim$ Bin(m, p). Show that $X + Y \sim$ Bin$(m + n, p)$.

[*Note:* If X, Y are integer-valued, independent and have prbability mass functions f and g, show that $X + Y$ has pmf $h(n) := \sum_{k \in \mathbb{Z}} f(k)g(n - k)$].

4. If $A \in \mathcal{B}(\mathbb{R}^2)$ has positive Lebesgue measure, show that for some $x \in \mathbb{R}$ the set $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$ has positive Lebesgue measure in $\mathbb{R}$.

5. Compute mean, variance and moments (as many as possible!) of the Normal(0,1), exponential(1), Beta(p,q) distributions.

6. (1) Suppose $X_n \geq 0$ and $X_n \to X$ a.s. If $\mathbf{E}[X_n] \to \mathbf{E}[X]$, show that $\mathbf{E}[|X_n - X|] \to 0$.

(2) If $\mathbf{E}[|X|] < \infty$, then $\mathbf{E}[|X|\mathbf{1}_{|X|>A}] \to 0$ as $A \to \infty$.

7. (**Differentiating under the integral**). Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, satisfy the following assumptions.

(1) $x \to f(x, \theta)$ is Borel measurable for each θ.

(2) $\theta \to f(x, \theta)$ is continuously differentiable for each x.

(3) $f(x, \theta)$ and $\frac{\partial f}{\partial \theta}(x, \theta)$ are uniformly bounded functions of (x, θ).

Then, justify the following "differentiation under integral sign" (including the fact that the integrals here make sense).

$$\frac{d}{d\theta} \int_a^b f(x, \theta) dx = \int_a^b \frac{\partial f}{\partial \theta}(x, \theta) \ dx$$

[**Hint:** Derivative is the limit of difference quotients, $h'(t) = \lim\limits_{\epsilon \to 0} \frac{h(t+\epsilon)-h(t)}{\epsilon}$].

8. Let X be a non-negative random variable.

(1) Show that $\mathbf{E}[X] = \int_0^\infty \mathbf{P}\{X > t\} dt$ (in particular, if X is a non-negative integer valued, then $\mathbf{E}[X] = \sum_{n=1}^\infty \mathbf{P}(X \geq n)$).

(2) Show that $\mathbf{E}[X^p] = \int_0^\infty p t^{p-1} \mathbf{P}\{X \geq t\} dt$ for any $p > 0$.

9. Let X be a non-negative random variable. If $\mathbf{E}[X]$ is finite, show that $\sum_{n=1}^\infty \mathbf{P}\{X \geq an\}$ is finite for any $a > 0$. Conversely, if $\sum_{n=1}^\infty \mathbf{P}\{X \geq an\}$ for some $a > 0$, show that $\mathbf{E}[X]$ is finite.

10. (1) If X, Y are independent random variables, show that $\text{Cov}(X, Y) = 0$.

(2) Give a counterexample to the converse by giving an infinite sequence of random variables $X_1, X_2, \ldots$ such that $\text{Cov}(X_i, X_j) = 0$ for any $i \neq j$ but such that X_i are not independent.

(3) Suppose $(X_1, \ldots, X_m)$ has (joint) normal distribution (see the first question). If $\text{Cov}(X_i, X_j) = 0$ for all $i \leq k$ and for all $j \geq k+1$, then show that $(X_1, \ldots, X_k)$ is independent of $(X_{k+1}, \ldots, X_m)$.

11. Let X be a non-negative random variable with all moments (i.e., $\mathbf{E}[X^p] < \infty$ for all $p < \infty$). Show that $\log \mathbf{E}[X^p]$ is a convex function of p.

12. Let $\Omega = \{1, 2, \ldots, n\}$. For a probability measure $\mathbf{P}$ on Ω, we define it "entropy" $H(\mathbf{P}) := -\sum_{k=1}^n p_k \log p_k$ where $p_k = \mathbf{P}\{k\}$ and it is understood that $x \log x = 0$ if $x = 0$. Show that among all probability measures on Ω, the uniform probability measure (the one with $p_k = \frac{1}{n}$ for each k) is the unique maximizer of entropy.

13. (1) Let ξ_1, ξ_2, ξ_3 be independent $\text{Ber}_{\pm}(1/2)$ random variables. Define $X_1 = \xi_2\xi_3$, $X_2 = \xi_1\xi_3$ and $X_3 = \xi_1\xi_2$. Show that X_1, X_2, X_3 are pairwise independent but not independent.

(2) Suppose $2 \le k < n$. Give an example of n random variables $X_1, \dots, X_n$ such that any subset of k of these random variables are independent but no subset of $k+1$ of them is independent.

(3) Suppose $(X_1, \dots, X_n)$ has a multivariate Normal distribution. Show that if X_i are pairwise independent, then they are independent.

14 (A quantitative characterization of absolute continuity). Suppose $\mu \ll \nu$. Then, show that given any $\epsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu(A) < \epsilon$. (The converse statement is obvious but worth noticing). [**Hint**: Argue by contradiction].

15. (1) If $\mu_n \ll \nu$ for each n and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \ll \nu$? If $\mu_n \perp \nu$ for each n and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \perp \nu$? In either case, justify or give a counterexample.

(2) Suppose X, Y are independent (real-valued) random variables with distribution μ and ν respectively. If μ and ν are absolutely continuous w.r.t Lebesgue measure, show that the distribution of $X + Y$ is also absolutely continuous w.r.t Lebesgue measure.

16. Suppose $(X_1, \dots, X_n)$ has density f (w.r.t Lebesgue measure on $\mathbb{R}^n$).

(1) If $f(x_1, \dots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable functions g_k, $k \le n$. Then show that $X_1, \dots, X_n$ are independent. (Don't assume that g_k is a density!)

(2) If $X_1, \dots, X_n$ are independent, then $f(x_1, \dots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable densities $g_1, \dots, g_n$.

17. Show that it is not possible to define uncountably many independent $\text{Ber}(1/2)$ random variables on the probability space $([0,1], \mathcal{B}, \lambda)$.

18. Let X_i, $i \ge 1$ be random variables on a common probability space. Let $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be a measurable function (with product sigma algebra on $\mathbb{R}^{\mathbb{N}}$ and Borel sigma algebra on $\mathbb{R}$) and let $Y = f(X_1, X_2, \dots)$. Show that the distribution of Y depends only on the joint distribution of $(X_1, X_2, \dots)$ and not on the original probability space. [**Hint:** We used this to say that if X_i are independent Bernoulli random variables, then $\sum_{i \ge 1} X_i 2^{-i}$ has uniform distribution on $[0,1]$, irrespective of the underlying probability space.]

19. Let $(\Omega_i, \mathcal{F}_i, \mathbf{P}_i)$, $i \in I$, be probability spaces and let $\Omega = \times_i \Omega_i$ with $\mathcal{F} = \otimes_i \mathcal{F}_i$ and $\mathbf{P} = \otimes_i \mathbf{P}_i$. If $A \in \mathcal{F}$, show that for any $\epsilon > 0$, there is a cylinder set B such that $\mathbf{P}(A \Delta B) < \epsilon$.

20 (Existence of Markov chains). Let S be a countable set (with the power set sigma algebra). Two ingredients are given: *A transition matrix*, that is, a function $p : S \times S \to [0, 1]$ be a function such that $p(x, \cdot)$ is a probability mass function on S for each $x \in S$. (1) *An initial distribution*, that is a probability mass function μ_0 on S.

For $n \geq 0$ define the probability measure ν_n on S^{n+1} (with the product sigma algebra) by

$$\nu_n(A_0 \times A_1 \times \ldots \times A_n) = \sum_{(x_0,\ldots,x_n) \in A_0 \times \ldots \times A_n} \mu_0(x_0) \prod_{j=0}^{n-1} p(x_j, x_{j+1}).$$

Show that ν_n form a consistent family of probability distributions and conclude that a Markov chain with initial distribution μ_0 and transition matrix p exists.