HOMEWORK 5: DUE 28TH OCT

SUBMIT THE FIRST FOUR PROBLEMS ONLY

1. (Chung-Erdös inequality).

(1) Let A_i be events in a probability space. Show that

$$\mathbf{P}\left\{\bigcup_{k=1}^{n} A_k\right\} \geq \frac{(\sum_{k=1}^{n}\mathbf{P}(A_k))^2}{\sum_{k,\ell=1}^{n}\mathbf{P}(A_k \cap A_\ell)}$$

(2) Place r_m balls in m bins at random and count the number of empty bins Z_m. Fix $\delta > 0$. If $r_m > (1+\delta)m\log m$, show that $\mathbf{P}(Z_m > 0) \to 0$ while if $r_m < (1-\delta)m\log m$, show that $\mathbf{P}(Z_m > 0) \to 1$.

2. Let ξ, ξ_n be i.i.d. random variables with $\mathbf{E}[\log_+ \xi] < \infty$ and $\mathbf{P}(\xi = 0) < 1$.

(1) Show that $\limsup\limits_{n\to\infty} |\xi_n|^{\frac{1}{n}} = 1 \; a.s.$

(2) Let c_n be (non-random) complex numbers. Show that the radius of convergence of the random power series $\sum_{n=0}^{\infty} c_n\xi_n z^n$ is almost surely equal to the radius of convergence of the non-random power series $\sum_{n=0}^{\infty} c_n z^n$.

3. Give example of an infinite sequence of pairwise independent random variables for which Kolmogorov's zero-one law fails.

4. Suppose $r = \lambda n$ balls are put into n boxes at random (more than one ball can go into a box). If N_n denotes the number of empty boxes, show that for any $\delta > 0$, as $n \to \infty$,

$$\mathbf{P}\left(\left|\frac{N_n}{n} - e^{-\lambda}\right| > \delta\right) \to 0$$

5. (Erdös-Renyi random graph model). Let $V_n = \{1, 2, \ldots, n\}$. Fix $0 < p \leq 1$. Let $X_{i,j}$, $1 \leq i < j \leq n$, be i.i.d. Ber(p) random variables. Consider the random graph $\mathcal{G}(n,p)$ whose vertex set is V_n and whose edges consist of pairs $\{i,j\}$ such that $X_{i,j} = 1$.

(1) Show that

$$\mathbf{P}\{\mathcal{G}(n,p) \text{ is not connected }\} \leq \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}(1-p)k(n-k).$$

(2) Show that $\mathbf{P}\{\mathcal{G}(n,p) \text{ is connected}\} \to 1$ as $n \to \infty$.