Homework 6: Due 13th Nov

Submit the first four problems only

1. Let X_n, Y_n, X, Y be random variables on a common probability space.

(1) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ (all r.v.s on the same probability space), show that $aX_n + bY_n \xrightarrow{P} aX + bY$ and $X_nY_n \xrightarrow{P} XY$. [**Hint:** You could try showing more generally that $f(X_n, Y_n) \to f(X, Y)$ for any continuous $f : \mathbb{R}^2 \to \mathbb{R}$.]

(2) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{d} Y$ (all on the same probability space), then show that $X_nY_n \xrightarrow{d} XY$.

2. Let X_n be i.i.d exponential(1) random variables.

(1) Find a sequence of numbers b_n converging to 0 so that $\limsup\, b_nX_n = 1$ a.s.

(2) Find a sequence of numbers a_n converging to $+\infty$ so that $\limsup \frac{M_n}{a_n} = 1$ a.s.

[**Remark:** You will need probabilities like $\mathbf{P}\{b_nX_n > t\}$ and $\mathbf{P}\{a_n^{-1}M_n > t\}$. Use the explicit density of exponential distribution to compute these probabilities].

3. Show that the sequence $\{X_n\}$ is tight if and only if $c_nX_n \xrightarrow{P} 0$ whenever $c_n \to 0$.

4. Let X_i be i.i.d. Cauchy random variables with density $\frac{1}{\pi(1+t^2)}$. Show that $\frac{1}{n}S_n$ fails the weak law of large numbers by completing the following steps.

(1) Show that $t\mathbf{P}\{|X_1| > t\} \to c$ for some constant c.

(2) Show that if $\delta > 0$ is small enough, then $\mathbf{P}\{|\frac{1}{n-1}S_{n-1}| \ge \delta\} + \mathbf{P}\{|\frac{1}{n-1}S_n| \ge \delta\}$ does not go to 0 as $n \to \infty$ [*Hint:* Consider the possibility that $|X_n| > 2\delta n$].

(3) Conclude that $\frac{1}{n}S_n$ does not converge in probability to 0. [*Extra:* With a little more effort, you can try showing that there does not exist deterministic numbers a_n such that $\frac{1}{n}S_n - a_n \xrightarrow{P} 0$].

5. Let X_n, Y_n, X, Y be random variables on a common probability space.

(1) Suppose that X_n is independent of Y_n for each n (no assumptions about independence across n). If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (U, V)$ where $U \overset{d}{=} X$, $V \overset{d}{=} Y$ and U, V are independent. Further, $aX_n + bY_n \xrightarrow{d} aU + bV$.

(2) Give counterexample to show that the previous statement is false if the assumption of independence of X_n and Y_n is dropped.

6. For $\mathbb{R}^d$-valued random vectors X_n, X, we say that $X_n \xrightarrow{P} X$ if $\mathbf{P}(\|X_n - X\| > \delta) \to 0$ for any $\delta > 0$ (here you may take $\|\cdot\|$ to denote the usual norm, but any norm on $\mathbb{R}^d$ gives the same definition).

(1) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, show that $(X_n, Y_n) \xrightarrow{P} (X, Y)$.

(2) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, show that $X_n + Y_n \xrightarrow{P} X + Y$ and $\langle X_n, Y_n \rangle \xrightarrow{P} XY$. **[Hint:** Show more generally that $f(X_n, Y_n) \xrightarrow{P} f(X, Y)$ for any continuous function f by using the previous problem for random vectors].

7. (1) If X_n, Y_n are independent random variables on the same probability space and $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and U, V are independent.

(2) If $X_n \xrightarrow{d} X$ and $Y_n - X_n \xrightarrow{P} 0$, then show that $Y_n \xrightarrow{d} X$.

8. (1) (**Skorokhod's representation theorem**) If $X_n \xrightarrow{d} X$, then show that there is a probability space with random variables Y_n, Y such that $Y_n \stackrel{d}{=} X_n$ and $Y \stackrel{d}{=} X$ and $Y_n \xrightarrow{a.s.} Y$. [**Hint:** Try to construct Y_n, Y on the canonical probability space $([0, 1], \mathcal{B}, \mu)$]

(2) If $X_n \xrightarrow{d} X$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, show that $f(X_n) \xrightarrow{d} f(X)$. [**Hint:** Use the first part]

9. Let $\{X_i\}_{i \in I}$ be a family of r.v on $(\Omega, \mathcal{F}, \mathbf{P})$.

(1) If $\{X_i\}_{i \in I}$ is uniformly integrable, then show that $\sup_i \mathbf{E}|X_i| < \infty$. Give a counterexample to the converse statement.

(2) Suppose $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function that goes to infinity and $\sup_i \mathbf{E}[|X_i| h(|X_i|)] < \infty$. Show that $\{X_i\}_{i \in I}$ is uniformly integrable. In particular, if $\sup_i \mathbf{E}[|X_i|^p] < \infty$ for some $p > 1$, then $\{X_i\}$ is uniformly integrable.