Problem set 5

Due date: **3rd Nov**

Submit the starred exercises only

Exercise 26 (*)**.** Let $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N_2\left(\begin{bmatrix} \mu \\ \nu \end{bmatrix}, \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_{2,2} \end{bmatrix}\right)$. Assume that $\sigma_{1,1}\sigma_{2,2} - \sigma_{1,2}^2 \neq 0$.

(1) Find the expectations, variances and covariance of X and Y.
(2) Find the conditional distribution of X given Y.

Exercise 27 (*)**.** Find the expectation and variance for the following distributions.

(1) Bin(n,p), Geo(p), Pois(λ).
(2) N$(0,1)$, Gamma(ν,λ), Beta(a,b) (in the latter two classes, (pay attention to the special case of exponential and uniform, respectively).

Exercise 28 (*)**.** (1) (**Alternate formula for expectation in terms of CDF**). Let X be a random variable with pmf or pdf $f(u)$. Let F be the CDF of X.

(a) Assume that $X \geq 0$. Then show that $\mathbf{E}[X] = \int_0^\infty (1-F(t))dt$. [**Hint:** First consider the case when X is integer valued and $\mathbf{P}(X=k) = p_k$. Write the two sides separately and check that they are equal].
(b) For a general random variable X write it as X_+ and X_- and use the previous part to deduce that $\mathbf{E}[X] = \int_0^\infty (1-F(t))dt + \int_{-\infty}^0 F(t)dt$ (assuming the expectation exists).
(c) Check this for various distributions like Geo(p), Exp(λ) etc.

(2) (**Alternate expression for the variance**). Let X and Y be independent random variables having the same distribution. Then show that $\text{Var}(X) = \frac{1}{2}\mathbf{E}[(X-Y)^2]$.

Exercise 29. Let $X_1,\ldots,X_n$ be independent random variables having Exp(1) distribution. Let $S_n = X_1 + \ldots + X_n$.

(1) Find the conditional distribution of X_1 given S_n.
(2) Find the conditional distribution of S_n given X_1.

Exercise 30. (1) Let X be a random variable with finite second moment. Define $f(a) = \mathbf{E}[(X-a)^2]$. Show that f is minimized uniquely at $a = \mathbf{E}[X]$. (**Hint:** Show that $f(a) = f(\mathbf{E}[X]) + (a - \mathbf{E}[X])^2$).

(2) Let X and Y be random variables with finite second moment. Let $f(a,b) = \mathbf{E}[(Y-(a+bX))^2]$. Show that f is uniquely minimized when $b = \frac{\text{Cov}(X,Y)}{\text{Var}(X)}$ and $a = \mathbf{E}[Y] - b\mathbf{E}[X]$.

Exercise 31. A bacterium gives rise to N other bacteria and dies, where N is a non-negative integer valued random variable. Each of the new bacteria (independently of each other) gives rise to offsprings (the number of offsprings of each bacterium has the same distribution as N).

The process starts with one bacterium. Find the expected number of bacteria in the second generation. [**Hint:** Use the iterated conditional expectation formula, $\mathbf{E}[X] = \mathbf{E}\left[\mathbf{E}[X \,\middle|\, Y]\right]$].