8. PHRAGMEN-LINDELÖF THEOREMS

The maximum modulus principle states that a holomorphic function f on a bounded domain attains its maximum on the boundary. This is not true for unbounded domains. For example, on the upper half-plane, e^{-z^2} is holomorphic, and on the boundary of the half-plane (i.e., on the real line) it is bounded by 1. However $e^{-(iy)^2} = e^{y^2}$ and thus it grows rapidly on the imaginary axis. In particular, it is not bound by 1 in the half-plane.

Phragmen-Lindelöf theorems are theorems that prove a maximum modulus theorem on certain unbounded domains (eg., sectors) under an extra assumption on the growth of the function inside the domain. We state one sample[7].

Theorem 1 (Phragmen-Lindelöf for the half-plane)**.** *Let f be continuous on $\overline{\mathbb{H}}$ and holomorphic in $\mathbb{H}$. Let $M(r) = \max\{|f(z)| : z \in \mathbb{H}, |z| = r\}$. If $|f(x)| \le 1$ for all $x \in \mathbb{R}$ and $\frac{1}{r}\log M(r) \to 0$ as $r \to \infty$, then $|f(z)| \le 1$ for all $z \in \mathbb{H}$.*

As a corollary, we can get a version of this for any sector in Corollary 2below. It is in that form that the theorem is stated in Thangavelu's book[8] and used to prove Hardy's theorem. The proof given there is short and succinct, but we shall take a more long-winded (and leave some loose ends!) but hope that it is more conceptual and gives some insight into the phenomenon.

Corollary 2. *[Phragmen-Lindelöf for a sector] Let $\alpha > \frac{1}{2}$ and let $\Omega_\alpha = \{z = re^{i\theta} : -\frac{\pi}{2\alpha} < \theta < \frac{\pi}{2\alpha}\}$. Suppose $f : \overline{\Omega_\alpha} \to \mathbb{C}$ is continuous, holomorphic in Ω_α. Assume that $|f(z)| \le 1$ for $z \in \partial\Omega_\alpha$ (i.e., $\arg z = \pm\pi/2\alpha$) and that $|f(z)| \le Ce^{|z|^\beta}$ for some $\beta < \alpha$ and $C < \infty$. Then, $|f(z)| \le 1$ for all $z \in \Omega_\alpha$.*

Proof of the corollary. Let $\mathbb{H}_+ = \{z : \mathrm{Re}(z) > 0\}$ be the right-half plane. Clearly, we can define $z \mapsto z^{1/\alpha}$ holomorphically on $\mathbb{H}_+$. It extends continuously to the boundary of $\mathbb{H}_+$, and maps $\mathbb{H}_+$ (or its closure) to Ω_α (or its closure) in a bijective manner. Let $g(z) = f(z^{1/\alpha})$, so that it is defined on $\overline{\mathbb{H}_+}$. Then $|g(iy)| \le 1$ for $y \in \mathbb{R}$ and $|g(z)| \le Ce^{|z|^{\beta/\alpha}}$. Thus, $\frac{1}{r}\log M_g(r) \le \frac{1}{r}(\log|C| + r^{\beta/\alpha}) \to 0$ as $r \to \infty$ since $\beta < \alpha$. This shows that g satisfies the conditions of the theorem (replace the upper half-plane by the right half-plane) and hence, $|g(z)| \le 1$ for all $z \in \mathbb{H}_+$. Thus $|f| \le 1$ on Ω_α. ■

Instead of jumping directly to the proof of the theorem, we explain the problem in general and take a short digression. We work with harmonic (and sub-harmonic) functions.

The general problem: Let Ω be a bounded region with piecewise smooth boundary. Let $A \subset \partial\Omega$ and let $B = \partial\Omega \setminus A$ (assume that A is nice, like an arc on the boundary). Let $u \in C(\overline{\Omega})$ be harmonic (or sub-harmonic) in Ω and assume that we have the bounds $u(z) \le M_1$ if $z \in A$ and $u(z) \le M_2$ if $z \in B$. What can we say about $u(z)$ for $z \in \Omega$.

Suppose $M_1 \le M_2$. The maximum principle says that $u \le M_2$ in Ω. But if M_1 is much smaller than M_2 and z is close to A (and far from B), we may expect to get a much better bound for $u(z)$ (the bound ought to be close to M_1).

Example 3. Let $\Omega = \mathbb{D}$ and let $A = \{e^{i\theta} : \theta \in [0,a]\}$. Then, by the Poisson-integral formula

$$u(z) = \int_0^a u(e^{i\theta})P(z,\theta)\frac{d\theta}{2\pi} + \int_a^{2\pi} u(e^{i\theta})P(z,\theta)\frac{d\theta}{2\pi}$$

$$\le M_1\left(\int_0^a P(z,\theta)\frac{d\theta}{2\pi}\right) + M_2\left(\int_a^{2\pi} P(z,\theta)\frac{d\theta}{2\pi}\right).$$

[7]The presentation here follows Ahfors' beautiful book *Conformal invariants*, except that I avoid the use of the word *Harmonic measure* which most students had not seen before.

[8]Sundaram Thangavelu, *An introduction to the uncertainty principle*, pages 18–22.

Recall that $P(z,\theta)d\theta/2\pi$ is a probability measure on the unit circle. It is mostly concentrated in the part of the circle close to z. Thus, if z is close to the interior of the arc A, then the above bound is $M_1(1-\delta)+M_2\delta$ for a small δ. This is naturally better than the trivial bound M_2.

How do we solve the problem for a general region Ω when we do not know the Poisson kernel? Here is the idea.

(1) Suppose we can find a harmonic function $h_{A,B}:\overline{\Omega}\to\mathbb{R}$ that is harmonic in Ω, equal to 0 on A, equal to 1 on B. We shall also require that $h_{A,B}$ is continuous on $\overline{\Omega}\setminus\partial A$ (in the example above, ∂A consists of the two end points of the arc A).

(2) Let $v(z)=M_1+(M_2-M_1)h_{A,B}(z)$. Then, v is harmonic in Ω, continuous on $\overline{\Omega}\setminus\partial A$ and $v(z)\ge u(z)$ for all $z\in\partial\Omega\setminus\partial A$. Appeal to the generalized maximum principle[9] and conclude that $u(z)\le v(z)$ for all $z\in\Omega$.

(3) To put everything together, if only we manage to find the function $h_{A,B}$, then we get the bound $u(z)\le M_1+(M_2-M_1)h_{A,B}(z)$ for all $z\in\Omega$.

We work out two cases. But before that a remark relating this to holomorphic functions.

Remark 4. If f is holomorphic on $\overline{\Omega}$, then $\log|f|$ is a *sub-harmonic* function (if f has no zeros in $\overline{\Omega}$, then it would be harmonic). It is a fact[10] that the generalized maximum principle holds for sub-harmonic function (caution: the minimum principle is false!). In particular, if u is sub-harmonic and h is harmonic and $u\le h$ on $\partial\Omega$ (with a finite number of exceptions), then $u\le h$ on Ω. In particular, all the above considerations hold even if u is sub-harmonic, in particular of $u=\log|f|$. Thus, for all $z\in\Omega$,

$$\log|f(z)|\le M_1+(M_2-M_1)h_{A,B}(z).$$

An annulus: Let $\Omega=\{z:R_1<|z|<R_2\}$ and let $A=R_1S^1$ (thus $B=R_2S^1$). In this case, it is easy to see that $\log|z|$ is a harmonic function in Ω, continuous to the boundary and equal to $\log R_1$ (respectively $\log R_2$) on the inner circle (respectively, the outer circle). Thus,

$$h_{A,B}(z)=\frac{\log|z|-\log R_1}{\log R_2-\log R_1}.$$

Suppose $|z|=s$, and write $\log s=\alpha\log R_1+(1-\alpha)\log R_2$ with $\alpha=\frac{\log R_2-\log s}{\log R_2-\log R_1}$. Then, the bound we have is

$$\begin{aligned}u(z)&\le M_1+(M_2-M_1)\frac{\log s-\log R_1}{\log R_2-\log R_1}\\&=\alpha M_1+(1-\alpha)M_2.\end{aligned}$$

When applied to holomorphic functions, we get

Theorem 5 (Hadamard's three circle theorem)**.** *Let f be holomorphic on an annulus $\Omega=\{z:R_1<|z|<R_2\}$ and let $M(r)=\max_{|z|=r}|f(z)|$. Then, $\log M(r)$ is a convex function of $\log r$.*

Proof. For any $s_1<s<s_2$ with $\log s=\alpha\log s_1+(1-\alpha)\log s_2$, we have the bound

$$\log|f(z)|\le\alpha\log M(s_1)+(1-\alpha)M(s_2)$$

for any z with $|z|=s$. Take maximum over z to get the conclusion. ∎

[9] Suppose Ω is a bounded region and $h:\overline{\Omega}\to\mathbb{R}$ is harmonic in Ω and $\limsup_{z\to\zeta}h(z)\le 0$ for all $\zeta\in\partial\Omega\setminus F$ where F is a finite subset of $\partial\Omega$. Then $h(z)\le M$ for all $z\in\Omega$.

[10] See Rudin's *Real and complex analysis*, for example.

A semi-disk: Let $\Omega = \{z : |z| < R \text{ and } \operatorname{Im} z > 0\}$ and let $A = (-R, R)$ and $B = \{Re^{i\theta} : 0 < \theta < \pi\}$. What is $h_{A,B}$ in this case?

For z in the upper half-plane, consider the angle subtended by $[-R, R]$ at z (i.e., the angle at z in the triangle with vertices $-R, z, R$. For $z \in B$, this angle is $\pi/2$ while for $z \in A$ (think of z approaching A from above) this angle is π. We may rescale this function to get $h_{A,B}(z) = \frac{2}{\pi}(\arg(z+R) - \arg(z-R) + \pi)$ where arg is a branch of the argument defined on $\mathbb{C} \setminus (-\infty, 0]$ and taking values in $(-\pi, \pi)$. As argument is a harmonic function (locally it is the imaginary part of $\log z$), we see that $h_{A,B}$ is harmonic.

Thus, if f is holomorphic on $\overline{\Omega}$ and $M(r) = \max\{|f(z)| : |z| = r, \operatorname{Im} z \geq 0\}$, and $m = \max\{|f(x)| : -R \leq x \leq R\}$, then we have the bound

$$\log|f(z)| \leq \log m + (\log M(r) - \log m) h_{A,B}(z).$$

Now we are ready to prove the Phragmen-Lindelöf theorem on the half-plane.

Proof. For any $R > 0$, from the previous bound (since $m = 1$), we get

$$\log|f(z)| \leq h_R(z) \log M(R)$$

where we write h_R to denote the explicit dependence on R.

Observe that if z is fixed and $R \to \infty$, then $\arg(z \pm R) = O(\frac{1}{R})$. Therefore, $h_R(z) = O(1/R)$. By the assumption that $\frac{1}{r} \log M(r) \to 0$ as $r \to \infty$, we see that $\log|f(z)| \leq 0$ or equivalently, $|f(z)| \leq 1$ for all $z \in \mathbb{H}$. ∎