9. Harmonic functions and Poisson integral formula

Let $\Omega \subseteq \mathbb{C}$ be a region (connected open set). A function $u : \Omega \mapsto \mathbb{R}$ is said to be *harmonic* if $u \in C^2(\Omega)$ and $\Delta u(z) = 0$ for all $z \in \Omega$. Here $\Delta u(z) := u_{x,x}(z) + u_{y,y}(z)$ with $z = x+iy$. We recall some basic properties of harmonic functions. We do not prove them here, but expect that you already know them[11].

(1) (**Mean value property**). Let $u \in C^2(\Omega)$. Then u is harmonic if and only if it has the mean value property (MVP). Recall that u is said to have MVP if $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta$ for all $z \in \Omega$ and $r > 0$ such that $\overline{\mathbb{D}}(z,r) \subseteq \Omega$. Here we have averaged u over a circle centered at z. One can also average over disks. In other words, MVP is equivalent to saying $u(z) = \frac{1}{\pi r^2}\int_{\mathbb{D}(z,r)} u(w)dm(w)$ for all $z \in \Omega$ and $r > 0$ such that $\overline{\mathbb{D}}(z,r) \subseteq \Omega$.

(2) (**Maximum principle**). Let Ω be a bounded region, $u \in C(\overline{\Omega})$ and $\Delta u = 0$ in Ω. Then $\max\limits_{\overline{\Omega}} u = \max\limits_{\partial\Omega} u$. Further, if $u(z) = \max\limits_{\overline{\Omega}} u$ for some $z \in \Omega$, then u must be constant. Since $-u$ is harmonic whenever u is, the same holds for the minimum too.

(3) (**Relationship to analytic functions**). If $f : \Omega \to \mathbb{C}$ is holomorphic, then $u = \operatorname{Re} f$ is harmonic in Ω. The converse is true if Ω is a disk, i.e., if Ω is a disk and u is harmonic on Ω, then there is a holomorphic function on Ω whose real part is equal to u.

It is good to understand the need for various conditions in the above facts.

Exercise 1. (1) What statements among the above change if Ω is not connected?

(2) Show that the maximum principle is false if Ω is not bounded. Take $\Omega = \mathbb{H}$, the upper half-plane.

(3) Give a region Ω and a harmonic function $u : \Omega \mapsto \mathbb{R}$ that is not the real part of a holomorphic function.

A useful computation: The following integral will be very important to us.

Lemma 2. *For all $z \in \mathbb{C}$ and any $r > 0$,*

$$\int_0^{2\pi} \log|z - re^{i\theta}| \frac{d\theta}{2\pi} = \log(|z| \vee r).$$

Proof. The function $z \mapsto \log|z|$ is harmonic on $\mathbb{C}\setminus\{0\}$ (why?). If $r < |z|$, then $\overline{\mathbb{D}(z,r)} \subseteq \mathbb{C}\setminus\{0\}$ and hence by MVP, we see that $\int_0^{2\pi} \log|z - re^{i\theta}| \frac{d\theta}{2\pi} = \log|z|$.

Now suppose $r > |z|$. Write $\log|z - re^{i\theta}| = \log|z| + \log r + \log|\frac{1}{z} - \frac{1}{r}e^{-i\theta}|$. Integrate over θ to get

$$\int_0^{2\pi} \log|z - re^{i\theta}| \frac{d\theta}{2\pi} = \log|z| + \log r + \int_0^{2\pi} \log\left|\frac{1}{z} - \frac{1}{r}e^{-i\theta}\right| \frac{d\theta}{2\pi}.$$

But $\left|\frac{1}{z}\right| < \frac{1}{r}$, hence by the previously considered case, the last integral is equal to $\log|1/z|$. Adding the first two terms we get $\int_0^{2\pi} \log|z - re^{i\theta}| \frac{d\theta}{2\pi} = \log r$.

This completes the proof when $|z| < r$ and $|z| > r$. When $|z| = r$, firstly convince yourself that the integral is well defined. Second, show that $r \mapsto \int_0^{2\pi} \log|z - re^{i\theta}| \frac{d\theta}{2\pi}$ is continuous in r. Conclude that the formula also holds for $|z| = r$. ■

[11]Rudin's *Real and complex analysis* has a whole chapter on harmonic functions and contains all the facts we mention.

Poisson integral formula: By the maximum principle, if u and v are continuous in $\overline{\Omega}$, harmonic in Ω, and equal to each other on $\partial\Omega$, then $u = v$. In other words, $u\,|_{\partial\Omega}$ determines u. But, is there an explicit way to recover u from its boundary values?

Let $\Omega = r\mathbb{D}$. Then, if $u \in C(r\overline{\mathbb{D}})$ and u is harmonic in $r\mathbb{D}$, then $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})P(z, re^{i\theta})d\theta$, where

$$P(z, re^{i\theta}) = \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} = \frac{r^2 - t^2}{r^2 + t^2 - 2rt\cos(\alpha - \theta)} \quad \text{if } z = te^{i\alpha} \text{ with } t < r.$$

For fixed r, the function $(z, \theta) \mapsto P(z, re^{i\theta})$ is called the *Poisson kernel* for $r\mathbb{D}$.

Exercise 3. Show that if $f : r\mathbb{T} \mapsto \mathbb{R}$ is continuous, then $u : r\mathbb{D} \to \mathbb{R}$ defined by $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})P(z, re^{i\theta})d\theta$ is harmonic in $r\mathbb{D}$ and $u(z_n) \to u(z_0)$ whenever $z_0 \in r\mathbb{T}$, $z_n \in r\mathbb{D}$ and $z_n \to z_0$.

The exercise is not trivial, but it is very much like many other proofs we have seen, for example Fejér's theorem or the proof of Fourier inversion etc. For fixed r, z, it may be seen that $d\mu(\theta) = P(z, re^{i\theta})\frac{d\theta}{2\pi}$ is a probability measure on $\mathbb{T}$. If $z_n \to re^{i\alpha}$ for some α, then this probability distribution concentrates closer and closer to the point $e^{i\alpha}$.

Another useful computation: We shall also need the following integral.

Lemma 4. *If $a \in r\overline{\mathbb{D}}$, then*

$$\int_0^{2\pi} \log|re^{i\theta} - a|\; P(a, re^{i\theta})\frac{d\theta}{2\pi} = \log(r^2 - |a|^2) - \log r.$$

Proof. Assume $|a| < r$ (then use continuity argument to get the result for $|a| = r$). The function $u(z) = \log|z - a|$ is harmonic on $\mathbb{C} \setminus \{a\}$ only, and hence Poisson integral cannot be applied directly (or we would get $u(a) = -\infty$!). We use a trick attributed to Kelvin, the so called *reflection principle*. The idea is to find a function v harmonic in a neighbourhood of $r\mathbb{D}$ such that $v = u$ on $r\mathbb{T}$. Then, the integral we want may also be written as $\int_0^{2\pi} v(re^{i\theta})\frac{d\theta}{2\pi}$ to which Poisson integral formula applies to give $v(a)$ as the answer!

What is v? Recall that if $|\zeta| < 1$, then the Mobius transformation $w \mapsto \frac{w - \zeta}{1 - \overline{\zeta}w}$ maps the unit circle to itself, i.e., $\frac{|w - \zeta|}{|1 - \overline{\zeta}w|} = 1$ if $|w| = 1$ and $|\zeta| < 1$. Apply this to $w = \frac{z}{r}$ with $|z| = r$ and $\zeta = \frac{a}{r}$ to get

$$1 = \frac{|\frac{z}{r} - \frac{a}{r}|}{1 - \frac{1}{r^2}\overline{a}z} = \frac{r|z - a|}{r^2 - \overline{a}z}.$$

Thus, if we set $v(z) = \log|r^2 - \overline{a}z| - \log r$, then $v(z) = u(z)$ whenever $|z| = r$. Further, v is harmonic on $\mathbb{C} \setminus \{\frac{r^2}{\overline{a}}\}$ which contains $r\overline{\mathbb{D}}$. We have found the v with the properties we wanted. Hence

$$\int_0^{2\pi} \log|re^{i\theta} - a|\frac{d\theta}{2\pi} = v(a) = \log(r^2 - |a|^2) - \log r.$$

This is what we wanted to show. ∎

A physicsly remark: From the inverse square law (Coulomb's law), the electric potential due to a unit charge at the origin of $\mathbb{R}^3$, is $\varphi(x) = \frac{1}{\|x\|}$ (a scalar field). Both for mathematics and physics, it makes sense to study the analogous laws in other dimensions. It turns out that in two dimensions, the correct analogue of Coulomb's law is inverse distance force, which means that the potential is $\varphi(z) = \log|z|$ (gradient of the potential should give the force). This is a useful way to sometimes interpret various manipulations above.

For example, what we did in the proof of Lemma 4 is this. First we considered $u(z) = \log|z-a|$, the potential due to a unit positive charge located at a. We replaced it by v, which is essentially the potential due to a unit charge placed at $\frac{r^2}{\bar{a}}$ (the "reflection" of a in the unit circle). By symmetry it turns out to give the same potential on the circle $r\mathbb{T}$ (This must be unclear since it is inaccurate, but think of a unit charge at $+i$ and a unit charge at $-i$. They both give the same potential on the real line. Our situation is analogous).

10. Poisson-Jensen formula

Lemma 5. *[Jensen's formula] Let f be a meromorphic function on the whole plane. If $f(0) \neq 0, \infty$, then, for any $r > 0$ we have*

$$\log|f(0)| = \int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi} - \sum_{i=1}^{m} \log \frac{r}{|a_i|} + \sum_{j=1}^{n} \log \frac{r}{|b_j|}$$

where $a_1, \ldots, a_m$ are the zeros of f in $r\overline{\mathbb{D}}$ and $b_1, \ldots, b_n$ are the poles of f in $r\overline{\mathbb{D}}$ (zeros and poles are always repeated according to multiplicity, thus a_is need not be distinct and the same for b_js).

Proof. Write $f(z) = g(z)\prod_{i=1}^{m}(z-a_i)\prod_{j=1}^{n}\frac{1}{z-b_j}$, where g is a meromorphic function on the whole plane that has no zeros or poles in $r\overline{\mathbb{D}}$. Then,

$$\log|f(z)| = \log|g(z)| + \sum_{i=1}^{m} \log|z-a_i| - \sum_{j=1}^{n} \log|z-b_j|.$$

Therefore,

$$\begin{aligned}
\int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi} &= \int_0^{2\pi} \log|g(re^{i\theta})| \frac{d\theta}{2\pi} + \sum_{i=1}^{m} \int_0^{2\pi} \log|re^{i\theta} - a_i| \frac{d\theta}{2\pi} - \int_0^{2\pi} \log|re^{i\theta} - b_j| \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \log|g(re^{i\theta})| \frac{d\theta}{2\pi} + m\log r - n\log r
\end{aligned}$$

by Lemma 2. Subtract this from $\log|f(0)|$ to get

$$\begin{aligned}
\log|f(0)| - \int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi} &= \log|g(0)| - \int_0^{2\pi} \log|g(re^{i\theta})| \frac{d\theta}{2\pi} + \sum_{i=1}^{m} \log|a_i| - m\log r - \sum_{j=1}^{n} \log|b_j| + n\log r \\
&= \log|g(0)| - \int_0^{2\pi} \log|g(re^{i\theta})| \frac{d\theta}{2\pi} - \sum_{i=1}^{m} \log \frac{r}{|a_i|} + \sum_{j=1}^{n} \log \frac{r}{|b_j|}.
\end{aligned}$$

As g is holomorphic in a neighbourhood of $r\mathbb{D}$ and has no zeros there, $z \to \log|g(z)|$ is harmonic in a neighbourhood of $r\mathbb{D}$. By MVP, $\log|g(0)| = \int_0^{2\pi} \log|g(re^{i\theta})| \frac{d\theta}{2\pi}$. Jensen's formula follows. ∎

We next prove a generalization of the Jensen's formula.

Lemma 6. *[Poisson-Jensen formula] Let f be a meromorphic function on the whole plane. If $r > 0$, $w \in r\mathbb{D}$, $f(w) \neq 0, \infty$, then,*

$$\log|f(w)| = \int_0^{2\pi} \log|f(re^{i\theta})| P(w, re^{i\theta}) \frac{d\theta}{2\pi} - \sum_{i=1}^{m} \log\left|\frac{r^2 - \bar{a}_i w}{r(w-a_i)}\right| + \sum_{j=1}^{n} \log\left|\frac{r^2 - \bar{b}_j w}{r(w-b_j)}\right|$$

where $a_1, \ldots, a_m$ are the zeros of f in $r\overline{\mathbb{D}}$ and $b_1, \ldots, b_n$ are the poles of f in $r\overline{\mathbb{D}}$.

Proof. As before, write $f(z) = g(z)\prod_{i=1}^m (z-a_i)\prod_{j=1}^n \frac{1}{z-b_j}$, where g is a meromorphic function on the whole plane that has no zeros or poles in $r\overline{\mathbb{D}}$. Then,

$$\log|f(z)| = \log|g(z)| + \sum_{i=1}^m \log|z-a_i| - \sum_{j=1}^n \log|z-b_j|.$$

Now put $z = re^{i\theta}$ and integrate again $P(w, re^{i\theta})$ over θ to get

$$\int_0^{2\pi} \log|f(re^{i\theta})|P(w,re^{i\theta})\frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} \log|g(re^{i\theta})|P(w,re^{i\theta})\frac{d\theta}{2\pi} + \sum_{i=1}^m \int_0^{2\pi} \log|re^{i\theta}-a_i|P(w,re^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi} \log|re^{i\theta}-b_j|\frac{d\theta}{2\pi}$$

By Lemma 4 we find the integrals in the two summands. Further, $\log|g|$ is harmonic in a neighbourhood of $r\mathbb{D}$, hence by the Poisson integral formula, $\int_0^{2\pi} \log|g(re^{i\theta})|P(w,re^{i\theta})\frac{d\theta}{2\pi} = \log|g(w)|$. We conclude

$$\int_0^{2\pi} \log|f(re^{i\theta})|P(w,re^{i\theta})\frac{d\theta}{2\pi} = \log|g(w)| + \sum_{i=1}^m [\log(r^2-|a_i|^2) - \log r] - \sum_{j=1}^n [\log(r^2-|b_j|^2) - \log r].$$

Subtract from $\log|f(w)| = \log|g(w)| + \sum_{i=1}^m \log|w-a_i| - \sum_{j=1}^n \log|w-b_j|$ to get

$$\log|f(0)| - \int_0^{2\pi} \log|f(re^{i\theta})|P(w,re^{i\theta})\frac{d\theta}{2\pi} = -\sum_{i=1}^m \log\frac{r^2-|a_i|^2}{r|w-a_i|} + \sum_{j=1}^n \log\frac{r^2-|b_j|^2}{r|w-b_j|}$$

which is what we wanted to show. ■

Even if f has a zero or pole at the origin, show that Jensen's formula can be modified as follows.

Exercise 7. Let f be meromorphic on the whole plane and suppose that $f(z) \sim c_f z^d$ as $z \to 0$ for some $d \in \mathbb{Z}$ and $c_f \in \mathbb{C}$. Then, show that

$$\log|c_f| = \int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} - \sum_{i=1}^m \log\frac{r}{|a_i|} + \sum_{j=1}^n \log\frac{r}{|b_j|}$$

where $a_1, \ldots, a_m$ are the zeros of f in $r\overline{\mathbb{D}}$ and $b_1, \ldots, b_n$ are the poles of f in $r\overline{\mathbb{D}}$ (if $d > 0$ then d many of the a_is are zero and if $d < 0$ then d many of the b_js are zero).