11. THE PRINCIPAL CHARACTERS IN NEVANLINNA THEORY

Let f be a meromorphic function on the whole plane. We define several quantities associated to it.

Counting function: Let $n_f(r)$ be the number of poles of f in $r\overline{\mathbb{D}}$. Set $N_f(r) = n_f(0)\log r + \int_0^r \frac{n_f(t)-n_f(0)}{t}dt$. If 0 is not a pole of f, then we can simply write $N_f(r) = \int_0^r \frac{n_f(t)}{t}dt$. In either case, it is a quantity we have seen before. Suppose $b_1, \ldots, b_n$ are the poles of f in $r\overline{\mathbb{D}}$, repeated with multiplicities. Check that $N_f(r) = \sum_{j=1}^n \log \frac{r}{|b_j|}$, the quantity that appeared in Jensen's formula.

Proximity function: Let $m_f(r) = \int_0^{2\pi} \log_+ |f(re^{i\theta})| \frac{d\theta}{2\pi}$. Here $\log_+ t = \log(t \vee 1) = (\log t) \vee 0$. In some sense, m_f measures how big f is (i.e, how close to ∞) on the circle $r\mathbb{T}$, on average. The positive part of log ensures that we suppress cancellation coming from positive and negative parts of the logarithm.

Nevanlinna characteristic function: Let $T_f(r) = m_f(r) + N_f(r)$. Note that both terms are non-negative, and so it T_f.

For other values: N_f counts the poles of f and m_f measures how close f is to infinity on $r\mathbb{T}$ (on average). Now we want to define similar quantities for any $a \in \mathbb{C}$, instead of $a = \infty$. To this end, define

$$n_f(r,a) = n_{\frac{1}{f-a}}(r), \;\; N_f(r,a) = N_{\frac{1}{f-a}}(r), \;\; m_f(r,a) = m_{\frac{1}{f-a}}(r) \;\text{ and }\; T_f(r,a) = T_{\frac{1}{f-a}}(r).$$

For example, $n_f(r,a)$ is the number of zeros of $f-a$ in $r\overline{\mathbb{D}}$, thus counting how many times the value a is attained. And $m_f(r,a) = \int_0^{2\pi} \log_+ \frac{1}{|f(re^{i\theta})-a|} \frac{d\theta}{2\pi}$ which measures proximity to a (on average on $r\mathbb{T}$).

Example 1. $f(z) = a_n z^n + \ldots + a_0$, a polynomial of degree n. Then, for large enough r we have $n_f(r) = n$. Therefore, $N_f(r) \sim n\log r$ as $r \to \infty$. Further, $f(z) \sim a_n z^n$, as $z \to \infty$. Hence, $m_f(r) \sim n\log r$, for $r \to \infty$. Then of course, $T_f(r) \sim 2n\log r$.

Example 2. $f(z) = e^z$. Then, $n_f(r) = 0$ for all r (hence $N_f(r) = 0$ too), while $m_f(r) = \int_0^{2\pi} (r\cos\theta)_+ \frac{d\theta}{2\pi} = \frac{r}{\pi}$. Thus, $T_f(r) = \frac{r}{\pi}$, with all the contribution coming from $m_f(r)$.

Now let $a \in \mathbb{C}$. If $a = 0$, we again get $T_f(r,0) = m_f(r) = \frac{r}{\pi}$ and $N_f(r,0) = 0$. If $a \neq 0$, the situation changes completely. The solutions to $e^z = a$ are all of the form $z_0 + 2\pi i n$, $n \in \mathbb{Z}$, for a specific z_0. Therefore, in $r\mathbb{D}$, there are approximately $\frac{r}{\pi}$ solutions (for large r). Thus $n_f(r,a) \sim \frac{r}{\pi}$ and then $N_f(r,a) \sim \frac{r}{\pi}$. But $m_f(r,a) = \int_0^{2\pi} \log_- |f(re^{i\theta}) - a| \frac{d\theta}{2\pi}$. We claim that $m_f(r,a) = O(1)$ as $r \to \infty$. Thus, $T_f(r,a) \sim \frac{r}{\pi}$ for $a \neq 0, \infty$ also, except that the entire contribution comes from $N_f(r,a)$ and almost nothing from $m_f(r)$!

Exercise 3. Do the same analysis for $f(x) = ze^z$.

Overview of what is to come in Nevanlinna theory: Let us keep as our goal Picard's theorem, which asserts that an entire function can miss at most one value in the complex plane. More generally, a meromorphic function misses at most two points in $\mathbb{C} \cup \{\infty\}$.

In Nevanlinna theory, a more elaborate study is made, not just whether a value is taken, but also how often (the counting function N_f measures this). For example, $z \mapsto e^z$ misses exactly two values, 0 and ∞. Two points may be noted about Nevanlinna's characteristic, counting and proximity functions.

(1) $T_f(r,a)$ is about the same, for all a (when r is large).

(2) Except for $a = 0, \infty$, in all other cases, the dominant contribution to $T_f(r,a)$ comes from $N_f(r,a)$.

This is illustrative of the general situation. Nevanlinna showed the following theorems (caution: the statements are given inaccurately now!) for a general non-constant meromorphic function.

(1) *First fundamental theorem of Nevanlinna theory:* For any $a \in \mathbb{C}$, we have that $T_f(r,a) - T_f(r) = O(1)$ as $r \to \infty$. Further $T_f(r) \to \infty$ as $r \to \infty$.

(2) *Second fundamental theorem of Nevanlinna theory:* For any distinct $a_1, \ldots, a_q$, we have

$$m_f(r) + \sum_{j=1}^{q} m_f(r, a_j) \le 2T_f(r) + \text{lower order terms}.$$

The second theorem implies that there are atmost two values of a for which $m_f(r,a) = T_f(r,a)$. Picard's theorem follows!

Some exercises: We shall actually make use of the following exercises.

First, about the function $\log_+$ which will appear often. The following estimates which will come in handy.

Exercise 4. For any $t, s > 0$, show that $\log_+(ts) \le \log_+ t + \log_+ s$ and $\log_+(t+s) \le \log_+ t + \log_+ s + \log 2$.

Secondly, about Nevanlinna's characteristic function.

Exercise 5. Let f be a non-constant meromorphic function.

(1) If $f = \frac{P}{Q}$ where P, Q are polynomials without common zeros and $d = \max\{\text{degree}(P), \text{degree}(Q)\}$, then $T_f(r) \sim d \log r$.

(2) If f is not a rational function, then $\frac{T_f(r)}{\log r} \to \infty$ as $r \to \infty$.

12. First fundamental theorem

Theorem 6 (First fundamental theorem of Nevanlinna theory)**.** *Let f be a non-constant meromorphic function on $\mathbb{C}$. Then, for any $a \in \mathbb{C}$, we have $T_f(r,a) = T_f(r) + O(1)$ as $r \to \infty$.*

The implied constants in $O(1)$ depend on f and a.

Proof. The idea is to simply rewrite Jensen's formula in terms of the functions introduced in the previous section. Thus, (assuming $f(0) \neq 0, \infty$),

$$\begin{aligned}
\log|f(0)| &= \int_0^{2\pi} \log_+ |f(re^{i\theta})| \frac{d\theta}{2\pi} + \sum_{j=1}^{n} \log \frac{r}{|b_j|} - \int_0^{2\pi} \log_+ \frac{1}{|f(re^{i\theta})|} \frac{d\theta}{2\pi} - \sum_{i=1}^{m} \log \frac{r}{|a_i|} \\
&= m_f(r) + N_f(r) - m_{1/f}(r) - N_{1/f}(r) \\
&= T_f(r) - T_{1/f}(r) \\
&= T_f(r) - T_f(r,0).
\end{aligned}$$

Thus we see that $T_f(r,0) = T_f(r) + O(1)$ as $r \to \infty$. We leave it to you to check that the last conclusion remains valid even if $f(0) = 0$ or $f(0) = \infty$ (see Exercise 7).

If $g(z) = f(z) - a$, then g has the same poles as f, while the zeros of g correspond to the poles of $1/(f-a)$. Therefore, $N_f(r) = N_g(r)$ and $N_f(r,a) = N_g(r,0)$. Further, $|\log_+|f(z)| - \log_+|g(z)|| \le \log 2 + \log_+|a|$. Thus, $|m_f(r) - m_g(r)| \le \log 2 + \log_+|a|$ while $m_f(r,a) = m_g(r,0)$. Thus, $|T_f(r) - T_g(r)| = O(1)$ as $r \to \infty$.

From the already proved case, $T_g(r,0) - T_g(r) = O(1)$ as $r \to \infty$. But clearly, $T_g(r) = T_f(r,a)$. Putting everything together, we see that $T_f(r,a) - T_f(r) = O(1)$ as $r \to \infty$. ∎

At this point I stopped typing as I have nothing to add to the excellent references easily available.

(1) Hayman's book *Meromorphic functions* (chapters 1 and 2), has everything I said in class (and more).

(2) Notes on the subject by Langley may be found at https://www.maths.nottingham.ac.uk/personal/jkl/pg1.pdf. This gives the same proof as we did in class (due to Nevanlinna himself), via the lemma on the logarithmic derivative.

(3) Notes on the subject by Eremenko are available at http://www.math.purdue.edu/~eremenko/dvi/weizmann.pdf. He gives a different proof due to Ahlfors, but also a short sketch of Nevanlinna's proof.