MIDTERM 1, TOPICS IN ANALYSIS

DATE: 25/SEP/2014

(1) **Solve any three problems. If you answer more, only the best three will be considered.**

(2) **The duration of the test is two hours.**

(3) **Each problem carries 10 marks.**

(4) **You may assume the results proved in class. But not the proofs!**

1. Consider the group $G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}$ with matrix multiplication. Find left and right Haar measures on G. [**Hint:** Identify G with $\mathbb{R}_+ \times \mathbb{R}$ and find the density of Haar measure with respect to Lebesgue measure on the latter space.]

2. Let G be a compact group. If μ is a left-Haar probability measure and ν is a right-Haar probability measure, show that $\mu = \nu$. Deduce that Haar measure on G is unique (up to multiplication by constants).

3. Let K be a bounded convex set in $\mathbb{R}^n$. Fix a unit vector $u \in \mathbb{R}^n$ and for $t \in \mathbb{R}$, define $K_t = \{x \in K : \langle x, u \rangle = t\}$. Let $I = \{t : |K_t| > 0\}$ and let $f : I \to \mathbb{R}$ be defined by $f(t) = |K_t|^{\frac{1}{n-1}}$. Show that I is an interval and that f is concave.

4. Suppose $(a_n)_n$ is a sequence of real numbers such that as $n \to \infty$,

$$\frac{1}{n} \sum_{k=1}^{n} e^{2\pi i \ell a_k} \to 0$$

for all *even numbers* $\ell \geq 2$. Show that a_n (mod 1) is equidistributed in $[0, 1]$. [**Remark:** Assume the theorems presented in class. Don't try to prove everything from scratch!]