MIDTERM 2, TOPICS IN ANALYSIS

DATE: 18/NOV/2014, 2:30-5:00 PM

(1) **Solve any number of problems. The maximum you can score is 45.**

(2) **Each problem carries 10 marks.**

(3) **You may assume the results proved in class except when asked explicitly to prove something!**

1. Let r be a positive integer. Show that

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|\sin(\pi r e^{i\theta})\right| d\theta = (2r+1)\log r - 2\log(r!) + \log \pi.$$

What is the answer if r is not an integer?

2. Let $A_{n\times n} = (a_{i,j})_{i,j\le n}$ be a matrix with $a_{i,j} \ge 0$ and $\sum_{j=1}^n a_{i,j} = 1$ for all i and $\sum_{i=1}^n a_{i,j} = 1$ for all j. Show that there is a permutation π of $\{1,2,\dots,n\}$ such that $a_{i,\pi(i)} > 0$ for all $i = 1,2,\dots,n$.

3. Let a_n, $n \ge 1$, be non-zero complex numbers such that $a_n \to \infty$ as $n \to \infty$. For $r > 0$, define $n(r) = \sum_{k=1}^{\infty} \mathbf{1}_{|a_k|\le r}$ and let $N(r) = \int_0^r \frac{n(t)}{t} dt$.

(1) Show that $N(r)$ is increasing in r and convex as a function of $\log r$.

(2) Show that $\frac{N(r)}{\log r} \to \infty$ as $r \to \infty$.

4. Let $f \in L^1(\mathbb{R})$ with Fourier transform $\hat{f}$. Let $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. For any $a > 0$, show that

$$\int_{\mathbb{R}} \hat{f}(\lambda)\varphi(a\lambda)e^{-i\lambda x} d\lambda = \frac{\sqrt{2\pi}}{a}\int_{\mathbb{R}} \varphi\left(\frac{t-x}{a}\right) f(t)dt$$

5. (1) Let P be a polynomial and let $f(x) = P(x)e^{-\frac{1}{2}x^2}$. Show that $\hat{f}(\lambda) = Q(\lambda)e^{-\frac{1}{2}\lambda^2}$ for some polynomial Q.

(2) When $P(x) = 2x^2 - 3x + 7$, find Q.