HOMEWORK 1, TOPICS IN ANALYSIS

DUE DATE: 16TH SEPTEMBER 2014

1. For any $f \in C[-1,1]$ (real-valued) and any $d \geq 1$, define $\tau_f(d) := \inf\{\|f-p\|_{\sup} : p \in \mathcal{P}_d\}$ where $\mathcal{P}_d$ is the space of (real) polynomials of degree d or less. Weierstrass' approximation theorem asserts that $\tau_f(d) \to 0$ as $d \to \infty$ but it is of interest to understand how well we can do with a given degree.

(1) Suppose $p \in \mathcal{P}_d$ is such that there exist $x_1 < x_2 < \ldots < x_{d+2}$ and $f(x_i) - p(x_i) = \pm\|f-p\|_{\sup}$ with the difference being positive for odd values of k and negative for even values of k (or vice versa). Then, show that p achieves the infimum in the definition of $\tau_f(d)$.

(2) Let $f(x) = x^{d+1}$. Show that $\tau_f(d) = 2^{-d}$ by considering the Chebyshev polynomials defined by $T_m(\cos\theta) = \cos(m\theta)$ (eg., $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1, \ldots$).

2. In this exercise we look at equidistribution in higher dimensions. Let $d \geq 1$ and let $(a_n)_n$ be a sequence in $\mathbb{R}^d$. We say that the sequence is equidistributed in $[0,1]^d$ if

$$\frac{1}{N}\#\{k \leq N : \overline{a_k} \in J\} \to |J|$$

for any rectangle $J = [a_1, b_1] \times \ldots \times [a_d, b_d]$. Note that $\overline{a}$ means that we take fractional parts co-ordinatewise.

(1) Show that $(a_n)_n$ is equidistributed if and only if $\frac{1}{N}\sum_{k=1}^{N} e(\ell \cdot a_k) \to 0$ for all $\ell \in \mathbb{Z}^d \setminus \{0\}$ (here $\ell \cdot a_k$ means the inner product between ℓ and a_k in $\mathbb{R}^d$).

(2) Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and let $\beta \in \mathbb{R}^d$. Define $a_k = k\alpha + \beta$ for $k \geq 1$. Show that $(a_n)_n$ is equidistributed if and only if $\alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$.

3. Two unrelated questions in measure theory.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = cx$ for all x with $c = f(1)$.

(2) H be a basis for the vector space $\mathbb{R}$ over the field $\mathbb{Q}$. Show that H is not Lebesgue measurable.