HOMEWORK 2, TOPICS IN ANALYSIS

DUE DATE: 23/OCT/2014

1. A matrix $A_{k\times n}$ with $k \le n$ is called a *Latin rectangle* if $a_{i,j} \in [n]$ for all i, j and no two entries in the same row or in the same column are equal.

Suppose $A_{k\times n}$ with $k \le n-1$ is a Latin rectangle. Use Hall's theorem to show that it is possible to extend it to a $(k+1)\times n$ Latin rectangle.

2. Let $A = (a_{i,j})_{i,j\le n}$ be a doubly-stochastic matrix. This means that $a_{i,j} \ge 0$ and $\sum_{j=1}^{n} a_j = 1$ for all i and $\sum_{i=1}^{n} a_{i,j} = 1$ for all j. Show that A can be written as a convex combination of permutation matrices.

3. Find the Fourier transforms of (a) $f(x) = \frac{1}{1+x^2}$ and (b) $f(x) = \frac{\sin^2(x)}{x^2}$, without resorting to contour integration. [**Hint:** Fourier inversion]

4. Show the analogue of Fejér's theorem for the Fourier transform as follows.

(1) Given $f \in L^1(\mathbb{R})$, define

$$f_T(x) = \frac{1}{2\pi}\int_{-T}^{T}\left(1 - \frac{|\lambda|}{T}\right)\hat{f}(\lambda)e^{i\lambda x}d\lambda$$

Show that $f_T \xrightarrow{L^1} f$ as $T \to \infty$. [**Hint:** Look back at the proof of injectivity of Fourier transform that we gave in class. Relace the Gaussian by an appropriate probability density].

(2) Let $\mathcal{A} = \{f \in L^1 : \hat{f} \text{ has compact support}\}$. The previous part shows that $\mathcal{A}$ is dense in L^1. Show that $\mathcal{A}$ is dense in $C_0(\mathbb{R})$ (continuous functions vanishing at infinity) endowed with the sup-norm.

5. Define the *Hermite functions* for $n \ge 0$ by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

Thus $h_0(x) = \frac{1}{\pi^{1/4}}e^{-\frac{1}{2}x^2}$ and $h_n(x) = H_n(x)e^{-\frac{1}{2}x^2}$ for some polynomial H_n of degree n. Show that $\hat{h_n}(\lambda) = (-i)^n h_n(\lambda)$. In other words, h_n are eigenfunctions of the Fourier transform (as an operator from $L^2(\mathbb{R})$ to itself).