TOPICS IN ANALYSIS - FINAL EXAMINATION

Give adequate reasons for your assertions but try to write succinctly. Each question carries 11 marks. Answer as many questions as you can. The maximum you can score is 50.

Problem 1. Let f be a non-constant meromorphic function on the complex plane.

(1) Show that $\int_0^{2\pi} N_f(r, e^{i\alpha})\frac{d\alpha}{2\pi} = T_f(r) - \log_+ |f(0)|$.

(2) Show that $\int_0^{2\pi} m_f(r, e^{i\alpha})\frac{d\alpha}{2\pi} \le \log 2$.

[*Hint:* Use the Poisson-Jensen formula $\int_0^{2\pi} \log |f(re^{i\theta}) - a| \frac{d\theta}{2\pi} = \log |f(0) - a| + N_f(r, a) - N_f(r, \infty)$.]

Problem 2. A professor of mathematics wants to dispose of his collection of n books. Hearing this, k students show up at his office and demand various books. Each student likes some of the books (and each student may like a different set of books). Assume that for any subset S of students, the number of books that at least one of them likes is at least $\sum_{i\in S} q_i$ (where $q_1, \ldots, q_k$ are some fixed positive integers).

Show that it is possible to give q_1 books to the first student, q_2 books to the second student, ..., q_k books to the kth student, in such a way that no student gets a book that she does not like.

Problem 3. Let G be a compact group and let μ be a left-Haar measure and let ν be a right-Haar measure. Show that $\mu = \nu$. Conclude that there is a unique bi-invariant probability measure on any compact group.

[*Hint:* Fix $f \in C(G)$ and consider the function $(x, y) \mapsto f(xy)$ on $G \times G$.]

Problem 4. Fix $n \ge 1$. Let $\mathcal{T}_k$ denote the set of all k-element subsets of $\{1, 2, \ldots, n\}$. If $k \le \frac{n-1}{2}$, then show that there is an injective function $f : \mathcal{T}_k \mapsto \mathcal{T}_{k+1}$ so that $f(A)$ is a superset of A for all $A \in \mathcal{T}_k$.

Problem 5. Show that the push-forward of the uniform measure on the unit sphere in $\mathbb{R}^3$ under the stereographic projection onto $\mathbb{C} \cup \{\infty\}$ is $d\mu(z) = \frac{1}{\pi(1+|z|^2)^2} dm(z)$ where m is the Lebesgue measure on the complex plane.

[*Remark:* You may choose to take the sphere to be tangential to the complex plane at the origin or to have it centered at the origin of the complex plane.]

Date: 22/April/2019, 2:00-5:00.