TOPICS IN ANALYSIS - MID TERM 1

Give adequate reasons for your assertions but try to write succinctly. Answer any four questions. Each question carries 11 marks. Answer as many questions as you can. The maximum you can score is 50.

Problem 1. An IISc mathematics professor evaluated $\sin(k)$ for $k = 1, 2, \ldots, 1000$ and obtained their histogram (shown below). He is very excited that he has discovered a new law of nature. How do you explain to him that the picture is a consequence of well-known mathematics that any student of this course knows (your explanation should include a precise statement of what the shape in the picture really is).

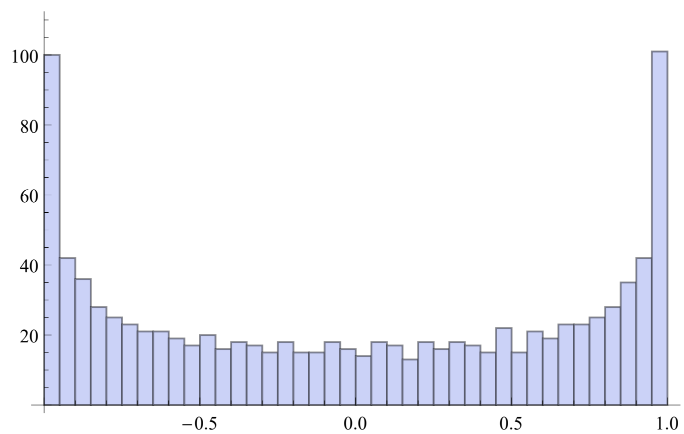

FIGURE 1. Histogram of the numbers $\sin(k)$, $1 \le k \le 1000$, in 50 bins of equal size (from -1 to 1).

Problem 2. Suppose $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is of the form $f(x, y) = e^{-\varphi(x,y)}$ where $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ is a convex function. Let $g(x) := \int_{\mathbb{R}} f(x, y)dy$. Show that $g(x) = e^{-\psi(x)}$ where $\psi : \mathbb{R} \mapsto \mathbb{R}$ is a convex function.

Problem 3. Let K be a bounded convex open set in $\mathbb{R}^n$. Let $K_t = K \cap \{x_1 = t\}$ be the t-section of K and let $|K_t|$ denote its $(n-1)$dimensional Lebesgue measure (measured inside $\{x_1 = t\}$ which is a copy of $\mathbb{R}^{n-1}$. Show that $t \mapsto |K_t|^{\frac{1}{n-1}}$ is concave and unimodal (unimodal means that there is some t_0 such that the function is increasing on $(-\infty, t_0]$ and decreasing on $[t_0, \infty)$).

Date: 24/February/2019, 2:00-4:15.

Problem 4. Let (X, d) be a compact metric space and let μ be a Borel probability measure on X. If K_n, K are compact subsets of X such that $K_n \to K$ in Hausdorff metric, show that $\mu(K) \geq \limsup_{n\to\infty} \mu(K_n)$.

Problem 5. Let $g : \mathbb{R} \mapsto \mathbb{R}_+$ be a non-negative integrable function such that $\int g(x)dx = 1$. Let $g_s(x) = \frac{1}{s}g(x/s)$ for $s > 0$. Show that $f \star g_s$ converges to f uniformly as $s \to 0$, for any bounded continuous function f. Would the conclusion be valid if we dropped the assumption that g is non-negative?