TOPICS IN ANALYSIS - MID TERM 2

Give adequate reasons for your assertions but try to write succinctly. Each question carries 12 marks. Answer as many questions as you can. The maximum you can score is 50.

Problem 1. Let B_n denote the number of set partitions of $\{1, 2, \dots, n\}$ and let $B_{n,k}$ denote the number of those with exactly k parts.

(1) Show that $B_n = e^{-1} \sum\limits_{k=0}^{\infty} \frac{k^n}{k!}$.

(2) Show that $\sum\limits_{k=1}^{n} B_{n,k} x(x-1) \dots (x-k+1) = x^n$ for all $x \in \mathbb{R}$.

Problem 2. Suppose $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a smooth function with a unique global minimum at the origin. Write an asymptotic expression for $I(\lambda) := \int_{\mathbb{R}^n} e^{-\lambda f(x)} dx$ as $\lambda \to \infty$. State your assumptions and give a brief justification (a full explanation with all details is not required, what you should explain is where the answer comes from).

Problem 3. (1) Let C_x denote the Ford circle at x for any $x \in \mathbb{Q}$. If $C_{p/q}$ touches $C_{r/s}$ then show that $C_{u/v}$ touches $C_{p/q}$ if and only if there is an $m \in \mathbb{Z}$ such that $u = r + mp$ and $v = s + mq$.

(2) Find an $SL_2(\mathbb{Z})$ matrix that takes $C_{2/3}$ to $C_{3/5}$.

Problem 4. Let $\Delta_n = \{(x_1, \dots, x_{n-1}) : x_i > 0$ for all i and $x_1 + \dots + x_{n-1} < 1\}$. Let μ_n denote the normalized Lebesgue measure on Δ_n (which is an open set in $\mathbb{R}^{n-1}$).

Let $\Pi_n(x) = nx_1$ denote the scaled projection from Δ_n onto the first co-ordinate. Show that $\mu_n \circ \Pi_n^{-1}$ converges to the exponential measure $e^{-x}dx$ on $(0, \infty)$.

Problem 5. Suppose $f : \mathbb{R} \mapsto \mathbb{C}$ is a function such that $\hat{f}$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$. Show that the data $\{f(k) : k \in \mathbb{Z}\}$ determines f within this class of functions. What if $\hat{f}$ is supported in $[-A, A]$?

[*Convention:* $\hat{f}(\lambda) = \int f(x)e^{-2\pi i x\lambda}dx$. You may make any reasonable assumptions on the 'niceness' of f and also assume the basic facts of Fourier analysis such as that the Fourier transform of a function determines the function.]

Date: 24/February/2019, 2:00-4:30.