Homework 1 (due 19/Aug/2012)

Try all the exercises. Submit only those marked with an asterisk (*).
Write the probability space before calculating any probabilities!

1. (*) Two positive integers $m \leq N$ are fixed. A box contains N coupons labelled $1, 2, \ldots, N$. A sample of m coupons is drawn.

(1) Write the probability space in the following two ways of drawing the sample.
   (a) (Sampling without replacement). A coupon is drawn uniformly at random, then a second coupon is drawn uniformly at random, and so on, till we have m coupons.
   (b) (Sampling with replacement). A coupon is drawn uniformly at random, its number is noted and the coupon is replaced in the box. Then a coupon is drawn at random from the box, the number is noted, and the coupon is returned to the box. This is done m times.

(2) Let $N = k + \ell$ where k, ℓ are positive integers. We think of $\{1, 2, \ldots, k\}$ as a sub-population of the whole population $\{1, 2, \ldots, N\}$. For each of the above two schemes of sampling (with and without replacement), calculate the probability that the sample of size m contains no elements from the sub-population $\{1, 2, \ldots, k\}$.

2. (*) (Feller, II.10.39) If r_1 indistinguishable red balls and r_2 indistinguishable blue balls are placed into n cells, find the number of distinguishable arrangements.

3. (*) A particular segment of the DNA in a woman is *ATTAGCGG* and the corresponding segment in her husband is *CTAAGGCG*. Write the probability space for the same DNA segment in the future child of this man-woman pair. Assume that all possible combinations are equally likely, and ignore the possibility of mutation.

4. (Feller, II.10.11) A man is given n keys of which only one fits his door. He tries them successively (sampling without replacement). The number of trials required may be $1, 2, \ldots, n$. Show that each of these outcomes has probability $1/n$.

5. (Feller, I.8.1) Among the digits 1,2,3,4,5 first one is chosen, and then a second selection is made among the remaining four digits. Assume that all twenty possible results have the same probability. Find the probability that an odd digit will be selected *(a)* the first time, *(b)* the second time, *(c)* both times.

6. In how many ways can two bishops be put on a chessboard so that they can take each other?

7. A deck of n cards labelled $1, 2, \ldots, n$ is shuffled well. Find the probability that the digits *(a)* 1 and 2, *(b)* 1, 2, and 3, appear as neighbours in the order named.

8. (Feller, II.10.8) What is the probability that among k digits *(a)* 0 does not appear; *(b)* 1 does not appear; *(c)* neither 0 nor 1 appears; *(d)* at least one of the two digits 0 or 1 does not appear? Let A and B represent the events in *(a)* and *(b)*. Express the other events in terms of A and B.

9. (Feller, II.10.28) A group of $2N$ boys and $2N$ girls is divided into two equal groups. Find the probability p that each group has equal number of boys and girls. Estimate p using Stirling's approximation.

10. (Feller, II.10.40) If r_1 dice and r_2 coins are thrown, how many results can be distinguished?

11. (Feller, II.12.1) Prove the following identities for $n \geq 2$. [*Convention:* Let n be a positive integer. Then $\binom{n}{y} = 0$ if y is not an integer or if $y > n$ or if $y < 0$].

$$1 - \binom{n}{1} + \binom{n}{2} - \ldots = 0$$
$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots = n2^{n-1}$$
$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \ldots = 0$$
$$2.1\binom{n}{2} + 3.2\binom{n}{3} + 4.3\binom{n}{4} + \ldots = n(n-1)2^{n-2}$$

12. (Feller, I.12.10) Prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$$

13. (Feller, I.12.20) Using Stirling's formula, prove that $\frac{1}{2^{2n}}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}$. [*Convention:* $a_n \sim b_n$ is shorthand for $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$].