Homework 2 (due 26/Aug/2012)

Try all the exercises. Submit only those marked with an asterisk (*).

1. (*) In each of the following cases you are given a countable set Ω and a function $f:\Omega\to\mathbb{R}$. Decide (with justification) whether $\sum_{n\in\Omega} f(\omega)$ converges absolutely or not.

(1) $\Omega=\mathbb{Z}$, $f(n)=e^{-\gamma|n|}$ where $\gamma\in\mathbb{R}$.
(2) $\Omega=\mathbb{N}$, $f(n)=n^{-\gamma}$ where $\gamma>0$. The answer depends on γ, of course.
(3) $\Omega=\mathbb{Z}\times\mathbb{Z}$, $f(n,m)=(m^2+n^2)^{-\gamma/2}$ where $\gamma>0$.

2. Let Ω be a set and let $\mathcal{P}(\Omega)$ be its power set (the set of all subsets of Ω). Show that there does not exist any onto function from $f:\Omega\to\mathcal{P}(\Omega)$ (or equivalently that there does not exist any one-one function from $g:\mathcal{P}(\Omega)\to\Omega$).

Use this to give another proof that the set of real numbers is uncountable.

3. Let $A_1,A_2,\ldots$ be countable subsets of a set Ω. Show that $\bigcup_n A_n$ is also countable (recall that the term countable includes finite sets).

4. Decide whether the following sets are countable or uncountable. The probability context in which these sets arise are mentioned in brackets (they are irrelevant to the solution of the problem).

(1) $\Omega=\{\underline{\omega}:\underline{\omega}=(\omega_1,\omega_2,\ldots),\ \omega_i=0 \text{ or } 1 \text{ for each } i\}$. [This is the sample space when a coin is tossed infinitely many times].
(2) $\Omega=\{\underline{\omega}:\underline{\omega}=(\omega_1,\omega_2,\ldots),\ \omega_i=0 \text{ or } 1 \text{ for each } i \text{ and only finite many } \omega_i \text{ are equal to } 1\}$.
[This is itself not a sample space for any particular experiment I can think of, but it essentially contains the sample spaces for many experiments that we shall consider. For example, "toss a coin till the seventh head", "toss a coin till you get five heads in succession", etc. Thus, from the above exercise, we immediately conclude that all these sample spaces are also countable].
(3) $\Omega=\{\underline{\omega}:\underline{\omega}=(\omega_1,\omega_2,\ldots,\omega_n),\ n\in\mathbb{N},\ \omega_i=0 \text{ or } 1 \text{ for each } i \text{ and } -10<\omega_1+\ldots+\omega_k<20 \text{ for } k<n, \text{ and } \omega_1+\ldots+\omega_n=-10 \text{ or } 20\}$.
[Two gamblers arrive with Rs.10 and Rs.20 in hand. They play a sequence of games in each of which the loser pays 1 rupee to the winner. They play till one of them becomes bankrupt. The sample space is essentially the above set Ω].
(4) Ω is the $1/3$-Cantor set. It is defined as follows. Let $K_0=[0,1]$. Remove the middle one-thirds $(1/3,2/3)$ to get $K_1=[0,1/3]\cup[2/3,1]$. In each interval, remove the middle one-thirds to get $K_2=[0,1/9]\cup[2/9,1/3]\cup[2/3,7/9]\cup[8/9,1]$. Continue to define K_n in the obvious manner. Then $K_0\supseteq K_1\supseteq K_2\ldots$. Define $\Omega=\bigcap_n K_n$. This is called the Cantor set. [**Hint:** Use base-3 expansion].

5. (*) Write the probability spaces for the following experiments. Coins and dice may not be fair!

(1) A coin is tossed till we get a head followed immediately by a tail. Find the probability of the event that the total number of tosses is at least N.
(2) A die is thrown till we see the number 6 turn up five times (not necessarily in succession). Find the probability that the number 1 is never seen.
(3) A coin is tossed till the first time when the number of heads (strictly) exceeds the number of tails. What is the probability that the number of tosses is at least 5.
(4) (Extra exercise for fun! Do not submit this part) In the previous experiment, find the probability that the number of tosses is more than N.